Approximating Riemann Mappings by Circle Packings

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Introduction

An artist travels to Italy and sees the Sistine Chapel for the first time. After seeing *The Last Judgment*, he is inspired to make a replica in the form of a stained glass window. However, he can only make windows in a circular form, and he would like to preserve nearly every aspect of Michelangelo's creation, including the angles. How is he going to do this?

Theoretically, the Riemann mapping theorem guarantees that this can be done. The theorem states that any bounded, hole-less domain in $\mathbb{C}$ is biholomorphically equivalent to the unit disc D. In fact, this map is angle preserving. However, many proofs of this theorem are non-constructive and give no intuition of how to even approximate this biholomorphic map.

To remedy this, William Thurston, at the Bieberbach conference in 1985, conjectured that circle packings, an arrangement of circles inside a given boundary such that no two overlap and all of them are mutually tangent, can be used to describe a discrete counterpart to the Riemann mapping theorem, namely, mapping circle packings inside a hole-less domain to circle packings inside the unit disk. What is beautiful about this observation is that circle packings appear in everyday life. From the bubbles of a Coca-Cola to the honeycombs of bees, circle packings seem like natural geometric objects for approximation. In fact, the 'honeycomb' packing is a pivotal object towards the end of this thesis.

This thesis is based on the sources [1] and [2]. We begin by proving various facts about hyperbolic geometry, which eventually lead to a road map in proving theorems about circle packings. We then develop the theory of quasiconformal maps, proving various analogous statements in complex analysis, like the Schwartz reflection principle, Montel's theorem, and Hurwitz's theorem. We end with proving the rigidity of the honeycomb packing, with a proof of the main theorem of this thesis, informally stated below.

Theorem 0.1 (Informal Rodin-Sullivan-Thurston Theorem)**.** *Let U be a simply connected domain with two distinct points z_0, z_1. Let $\phi : U \to D$ be the unique conformal map such that $\phi(z_0) = 0$ and $\phi(z_1) > 0$ (as guaranteed by the Riemann mapping theorem). For small $\varepsilon > 0$, let $\mathcal{C}_\varepsilon$ be the portion of the honeycomb packing of circles of radius ε that are contained in U, and let $\mathcal{C}'_\varepsilon$ be a packing of D 'combinatorially equivalent' to $\mathcal{C}_\varepsilon$ where the boundary circles are tangent to the unit circle. This gives rise to approximate maps $\phi_\varepsilon : U_\varepsilon \to D_\varepsilon$ where $U_\varepsilon \subset U$ and $D_\varepsilon \subset D$ are regions that converge to U and D respectively. After normalizing $\phi_\varepsilon(z_0) = 0$ and $\phi_\varepsilon(z_1) > 0$, we have $\phi_\varepsilon \to \phi$ as $\varepsilon \to 0$ uniformly on compact subsets of U (also known as 'local uniform' convergence).*

1. Hyperbolic Geometry

We first define an inner product on $D := \{z \in \mathbb{C} \,;\, |z| < 1\}$.

Definition 1.1. Fix $w \in D$. Then for $z_1, z_2 \in D$, define the *hyperbolic Riemannian metric* $\langle \cdot, \cdot \rangle_w : D^2 \to \mathbb{R}$ as

$$\langle z_1, z_2 \rangle_w := \frac{4\langle z_1, z_2 \rangle_{\text{euc.}}}{(1 - |w|^2)^2}$$

It is easy to see that $\langle \cdot, \cdot \rangle_w$ is an inner product for all fixed $w \in D$. Note that the norm associated to this inner product is

$$\|z\|_w := \frac{2|z|}{1 - |w|^2}.$$

We will use this in the next definition of length.

Definition 1.2. Let $\gamma : [0, 1] \to D$ be a piecewise-C^1 curve. Recall that this means γ is continuous on $[0, 1]$ and if there exists points

$$0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1,$$

where $\gamma'(t)$ exists and is continuous on $[x_k, x_{k+1}]$ (where the derivative at the endpoints is defined to be the one-sided limits). The *hyperbolic length of γ* is

$$\text{length}_{\text{hyp.}}(\gamma) := \int_0^1 \|\gamma'(t)\|_{\gamma(t)} dt = \int_0^1 \frac{2|\gamma'(t)|}{1 - |\gamma(t)|^2} dt \geq 0.$$

It is easy to see that hyperbolic length is invariant under complex conjugation, reparametrization and reversing orientation. It is also invariant under Möbius transformations of the disk by the Schwartz-Pick lemma, stated below.

Theorem 1.3 (Schwartz-Pick lemma)**.** *Let $f : D \to D$ be holomorphic. Then for all $z, w \in D$ we have*

$$\left| \frac{f(z) - f(w)}{1 - \overline{f(z)} f(w)} \right| \leq \left| \frac{z - w}{1 - \bar{z} w} \right|$$

and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$$

where both inequalities have equality if and only if f is a Möbius transformation.

The proof of this theorem can be found on page 172 of [3].

Definition 1.4. The *hyperbolic metric on D* is the function $d_{\text{hyp.}} : D^2 \to \mathbb{R}_{\geq 0}$ defined by

$$d_{\text{hyp.}}(z_1, z_2) := \inf_\gamma \text{length}_{\text{hyp.}}(\gamma)$$

where the infimum is taken over piecewise-C^1 curves $\gamma : [0, 1] \to D$ where $\gamma(0) = z_1$ and $\gamma(1) = z_2$.

Proposition 1.5. *$d_{\text{hyp.}}(\cdot, \cdot)$ is indeed a metric.*

Proof. $\underline{d_{\text{hyp.}} \geq 0}$: This is clear as $d_{\text{hyp.}}$ is defined as the infimum of nonnegative numbers.

$\underline{d_{\text{hyp.}}(z_1, z_2) = 0 \iff z_1 = z_2}$: The if direction is clear by considering the constant curve. For the only if direction, we prove the contrapositive. If $z_1 \neq z_2$, then

$$0 < |z_1 - z_2| = \inf_{\gamma} \int_{\gamma} |dz| \leq \inf_{\gamma} \int_{\gamma} \frac{2}{1 - |z|^2} |dz|$$

where the infimum is taken over piecewise-C^1 curves $\gamma : [0, 1] \to D$ where $\gamma(0) = z_1$ and $\gamma(1) = z_2$. Thus $d_{\text{hyp.}}(z_1, z_2) \neq 0$.
$\underline{d_{\text{hyp.}}(z_1, z_2) = d_{\text{hyp.}}(z_2, z_1)}$: This is easy to see as $\text{length}_{\text{hyp.}}(\cdot)$ is invariant under reversing orientation.
$\underline{d_{\text{hyp.}}(z_1, z_3) \leq d_{\text{hyp.}}(z_1, z_2) + d_{\text{hyp.}}(z_2, z_3)}$: Let $\gamma_0 : [0, 1] \to D$ be an arbitrary piecewise-C^1 curve such that $\gamma_0(0) = z_1, \gamma_0(1) = z_2$, and let $\gamma_1 : [0, 1] \to D$ be an arbitrary piecewise-C^1 curve such that $\gamma_1(0) = z_2, \gamma_0(1) = z_3$. Furthermore, we define $\gamma : [0, 1] \to D$ by

$$\gamma(t) \coloneqq \begin{cases} \gamma_0(2t) & t \in [0, 1/2) \\ \gamma_1(2t - 1) & t \in [1/2, 1] \end{cases}.$$

It is easy to see that γ is piecewise-C^1, $\gamma(0) = z_1$ and $\gamma(1) = z_3$. Furthermore,

$$d_{\text{hyp.}}(z_1, z_3) \leq \text{length}_{\text{hyp.}}(\gamma) = \text{length}_{\text{hyp.}}(\gamma_0) + \text{length}_{\text{hyp.}}(\gamma_1).$$

Taking the infimum over piecewise-C^1 curves $\gamma_0 : [0, 1] \to D$ where $\gamma_0(0) = z_1$ and $\gamma_0(1) = z_2$ and then taking the infimum over piecewise-C^1 curves $\gamma_1 : [0, 1] \to D$ where $\gamma_1(0) = z_2$ and $\gamma_1(1) = z_3$ on both sides, we get the result. $\square$

Proposition 1.6. *Let $f : D \to D$ be holomorphic. Then*

$$d_{\text{hyp.}}\big(f(z_1), f(z_2)\big) \leq d_{\text{hyp.}}(z_1, z_2).$$

If f is a Möbius transformation, then we have equality.

Proof. Let $\gamma : [0, 1] \to D$ be a piecewise-C^1 curve where $\gamma(0) = z_1$ and $\gamma(1) = z_2$. Then

$$\begin{aligned} d_{\text{hyp.}}\big(f(z_1), f(z_2)\big) &\leq \int_{f \circ \gamma} \frac{2}{1 - |z|^2} |dz| \\ &= \int_0^1 |\gamma'(t)| \frac{2|f'\big(\gamma(t)\big)|}{1 - |f\big(\gamma(t)\big)|^2} dt \\ &\leq \int_0^1 \frac{2|\gamma'(t)|}{1 - |\gamma(t)|^2} dt \end{aligned}$$

where the second inequality follows from the Schwartz-Pick lemma. Taking the infimum over piecewise-C^1 curves $\gamma : [0, 1] \to D$ where $\gamma(0) = z_1$ and $\gamma(1) = z_2$ on both sides, we obtain the inequality. For the equality, we use the equality statement of the Schwartz-Pick lemma, as well as realize that for every piecewise-C^1 curve $r : [0, 1] \to D$ with $r(0) = f(z_1)$ and $r(1) = f(z_2)$ there exists a piecewise-C^1 curve $\gamma : [0, 1] \to D$ such that $r = f \circ \gamma$ (take $\gamma := f^{-1} \circ r$). $\square$

Proposition 1.7. *For all $0 \leq x \leq y < 1$, we have*

$$d_{\text{hyp.}}(x, y) = \log \frac{(1 + y)(1 - x)}{(1 - y)(1 + x)}.$$

Furthermore, if $\gamma : [0,1] \to D$ is a piecewise-C^1 curve such that $\gamma(0) = x$, $\gamma(1) = y$, and $\text{length}_{\text{hyp.}}(\gamma) = d_{\text{hyp.}}(x,y)$*, then* $\text{image}(\gamma) = [x,y]$*. Additionally, such a curve exists.*

Proof. Let $\gamma : [0,1] \to D$ be a piecewise-C^1 curve where $\gamma(0) = x$ and $\gamma(1) = y$. Then

$$\begin{aligned}\int_0^1 \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2}dt &\geq \int_0^1 \frac{2\big|\Re\big(\gamma'(t)\big)\big|}{1-|\gamma(t)|^2}dt \geq \int_0^1 \frac{2\big|\Re\big(\gamma'(t)\big)\big|}{1-\Re\big(\gamma(t)\big)^2}dt \geq \int_0^1 \frac{2\Re\big(\gamma(t)\big)'}{1-\Re\big(\gamma(t)\big)^2}dt \\ &= \int_x^y \frac{2}{1-t^2}dt = \log\frac{1+y}{1-y} - \log\frac{1+x}{1-x} = \log\frac{(1+y)(1-x)}{(1-y)(1+x)}.\end{aligned}$$

Note that all the inequalities above are all equal if and only if $\Im(\gamma) = 0$ and $\Re(\gamma)' \geq 0$. If we consider $\gamma(t) := (y-x)t + x$, we satisfy these conditions as well as the original curve conditions. If γ is a piecewise-C^1 curve such that $\gamma(0) = x$, $\gamma(1) = y$, and $\text{length}_{\text{hyp.}}(\gamma) = d_{\text{hyp.}}(x,y)$, we see that from the above discussion we have $\Im(\gamma) = 0$ and $\Re(\gamma)' \geq 0$. Thus $\Re(\gamma)$ is nondecreasing, and this implies $\text{image}(\gamma) = [x,y]$. $\square$

Corollary 1.8. *For all $z_1, z_2 \in D$, we have*

$$d_{\text{hyp.}}(z_1,z_2) = \log\frac{1+\left|\frac{z_2-z_1}{1-\overline{z_1}z_2}\right|}{1-\left|\frac{z_2-z_1}{1-\overline{z_1}z_2}\right|} = 2\operatorname{arctanh}\left|\frac{z_2-z_1}{1-\overline{z_1}z_2}\right|.$$

Furthermore, if $\gamma : [0,1] \to D$ is a piecewise-C^1 curve such that $\gamma(0) = z_1$, $\gamma(1) = z_2$, and $\text{length}_{\text{hyp.}}(\gamma) = d_{\text{hyp.}}(z_1,z_2)$*, then* $\text{image}(\gamma) = \varphi_{z_1,z_2}^{-1}\big([0,|\frac{z_2-z_1}{1-\overline{z_1}z_2}|]\big)$ *where* $\varphi_{z_1,z_2}(z) := \frac{z-z_1}{1-\overline{z_1}z}e^{-i\operatorname{Arg}(\frac{z_2-z_1}{1-\overline{z_1}z_2})}$*. Additionally, such a curve exists.*

Proof. Note $d_{\text{hyp.}}(z_1,z_2) = d_{\text{hyp.}}(0,|\frac{z_2-z_1}{1-\overline{z_1}z_2}|)$ by Proposition 1.6 and using the transformation φ_{z_1,z_2}. Suppose $\gamma : [0,1] \to D$ is a piecewise-C^1 curve such that $\gamma(0) = z_1$, $\gamma(1) = z_2$, and $\text{length}_{\text{hyp.}}(\gamma) = d_{\text{hyp.}}(z_1,z_2)$, then

$$\text{length}_{\text{hyp.}}(\varphi_{z_1,z_2}\circ\gamma) = \text{length}_{\text{hyp.}}(\gamma) = d_{\text{hyp.}}(z_1,z_2) = d_{\text{hyp.}}\left(0,\left|\frac{z_2-z_1}{1-\overline{z_1}z_2}\right|\right).$$

Thus $\text{image}(\varphi_{z_1,z_2}\circ\gamma) = [0,|\frac{z_2-z_1}{1-\overline{z_1}z_2}|]$ by Proposition 1.7. A curve that does this is given by $\varphi_{z_1,z_2}^{-1}\circ\gamma$ where $\gamma(t) = t|\frac{z_2-z_1}{1-\overline{z_1}z_2}|$ defined for $t \in [0,1]$. $\square$

Corollary 1.9. *Let $z_1, z_2, z_3 \in D$. Then*

$$d_{\text{hyp.}}(z_1,z_3) = d_{\text{hyp.}}(z_1,z_2) + d_{\text{hyp.}}(z_2,z_3)$$

if and only if $z_2 \in \varphi_{z_1,z_3}^{-1}\big([0,|\frac{z_3-z_1}{1-\overline{z_1}z_3}|]\big)$.

Proof. $\underline{\Longleftarrow}$: Suppose $\varphi_{z_1,z_3}(z_2) \in [0,|\frac{z_3-z_1}{1-\overline{z_1}z_3}|]$. Then

$$\begin{aligned}&d_{\text{hyp.}}\big(0,\varphi_{z_1,z_3}(z_2)\big) + d_{\text{hyp.}}\left(\varphi_{z_1,z_3}(z_2),\left|\frac{z_3-z_1}{1-\overline{z_1}z_3}\right|\right) \\ &\qquad\overset{1.7}{=} \log\frac{1+\varphi_{z_1,z_3}(z_2)}{1-\varphi_{z_1,z_3}(z_2)} + \log\frac{\big(1+|\frac{z_3-z_1}{1-\overline{z_1}z_3}|\big)\big(1-\varphi_{z_1,z_3}(z_2)\big)}{\big(1-|\frac{z_3-z_1}{1-\overline{z_1}z_3}|\big)\big(1+\varphi_{z_1,z_3}(z_2)\big)} \\ &\qquad= \log\frac{1+|\frac{z_3-z_1}{1-\overline{z_1}z_3}|}{1-|\frac{z_3-z_1}{1-\overline{z_1}z_3}|} = d_{\text{hyp.}}\left(0,\left|\frac{z_3-z_1}{1-\overline{z_1}z_3}\right|\right).\end{aligned}$$

Apply Proposition 1.6 to all distances above with the transformation φ_{z_1,z_3}^{-1} we get

$$d_{\text{hyp.}}(z_1, z_3) = d_{\text{hyp.}}(z_1, z_2) + d_{\text{hyp.}}(z_2, z_3).$$

$\underline{\Longrightarrow}$: We prove the contrapositive. Suppose $\varphi_{z_1,z_3}(z_2) \notin [0, |\frac{z_3-z_1}{1-\overline{z_1}z_3}|]$. Let $\gamma_0 : [0,1] \to D$ be a piecewise-C^1 curve such that $\gamma_0(0) = 0$ and $\gamma_0(1) = \varphi_{z_1,z_3}(z_2)$ satisfying $\text{length}_{\text{hyp.}}(\gamma_0) = d_{\text{hyp.}}\big(0, \varphi_{z_1,z_3}(z_2)\big)$ per Corollary 1.8. Similarly, let $\gamma_1 : [0,1] \to D$ be a piecewise-C^1 curve such that $\gamma_1(0) = \varphi_{z_1,z_3}(z_2)$ and $\gamma_1(1) = |\frac{z_3-z_1}{1-\overline{z_1}z_3}|$ satisfying $\text{length}_{\text{hyp.}}(\gamma_0) = d_{\text{hyp.}}\big(\varphi_{z_1,z_3}(z_2), |\frac{z_3-z_1}{1-\overline{z_1}z_3}|\big)$ per Corollary 1.8. Now we define $\gamma : [0,1] \to D$ by

$$\gamma(t) := \begin{cases} \gamma_0(2t) & t \in [0, 1/2) \\ \gamma_1(2t-1) & t \in [1/2, 1] \end{cases}.$$

It is easy to see that γ is piecewise-C^1, $\gamma(0) = 0$ and $\gamma(1) = |\frac{z_3-z_1}{1-\overline{z_1}z_3}|$. Furthermore,

$$\begin{aligned} d_{\text{hyp.}}\big(0, \varphi_{z_1,z_3}(z_2)\big) + d_{\text{hyp.}}&\left(\varphi_{z_1,z_3}(z_2), \left|\frac{z_3 - z_1}{1 - \overline{z_1}z_3}\right|\right) \\ &= \text{length}_{\text{hyp.}}(\gamma_0) + \text{length}_{\text{hyp.}}(\gamma_1) \\ &= \text{length}_{\text{hyp.}}(\gamma) > d_{\text{hyp.}}\left(0, \left|\frac{z_3 - z_1}{1 - \overline{z_1}z_3}\right|\right), \end{aligned}$$

where the inequality follows from Proposition 1.6 since $\text{image}(\gamma) \neq [0, |\frac{z_3-z_1}{1-\overline{z_1}z_3}|]$ (consider $\varphi_{z_1,z_3}(z_2)$). We finish by applying Proposition 1.6 to all distances above with the transformation φ_{z_1,z_3}^{-1} we get

$$d_{\text{hyp.}}(z_1, z_3) < d_{\text{hyp.}}(z_1, z_2) + d_{\text{hyp.}}(z_2, z_3). \qquad \square$$

Note that Corollary 1.9 can be used to compute relations between Euclidean and hyperbolic circles.

Example 1.10. For any $z \in D$, $r > 0$, and $0 < r' < 1$, we have

$$d_{\text{hyp.}}(0, z) = r \iff |z| = \tanh(\tfrac{r}{2}), \quad \text{and} \quad |z| = r' \iff d_{\text{hyp.}}(0, z) = 2\,\text{arctanh}(r').$$

Thus every hyperbolic circle with center at 0 is a Euclidean circle with center at 0 (and vice-versa). Similarly, every hyperbolic circle with center at $a \in D$ and radius $r > 0$ is a Euclidean circle because if $T(z) := \frac{z-a}{1-\overline{a}z}$, we have

$$\{z \mid d_{\text{hyp.}}(z, a) = r\} = \{z \mid d_{\text{hyp.}}(T(z), 0) = r\} = T^{-1}\big(\{z \mid d_{\text{hyp.}}(z, 0) = r\}\big).$$

Since T^{-1} sends Euclidean circles to Euclidean circles, we have the result. Now if we would like to know the Euclidean center and radius of a hyperbolic circle with center at $a \in D \setminus \{0\}$ and radius $r > 0$, we first note

$$\{z \mid d_{\text{hyp.}}(z, |a|) = r\} = \overline{\{z \mid d_{\text{hyp.}}(z, |a|) = r\}}$$

since $d_{\text{hyp.}}(\cdot, \cdot)$ is invariant under complex conjugates. Thus the imaginary part of the Euclidean center is 0 because the circle is symmetric about the real axis. Now if $z \in D$ such that $\Im(z) = 0$ and $d_{\text{hyp.}}(z, |a|) = r$, then

$$\tanh(\tfrac{r}{2}) = \frac{\big|z - |a|\big|}{1 - |a|z} \implies z = \frac{|a| \pm \tanh(\frac{r}{2})}{1 \pm |a| \tanh(\frac{r}{2})}.$$

Now the Euclidean center is thus the midpoint of these points. Doing the arithmetic, we see that the center is

$$c = |a|\frac{1-\tanh^2(\frac{r}{2})}{1-|a|^2\tanh^2(\frac{r}{2})}.$$

To get the radius, we take the larger of the z's and subtract c. We get that

$$r_{\text{euc.}} = \tanh(\tfrac{r}{2}) \cdot \frac{1-|a|^2}{1-|a|^2\tanh^2(\frac{r}{2})}. \tag{1.11}$$

To get the original circles Euclidean center, we note

$$e^{i\theta}\{z \mid d_{\text{hyp.}}(z,|a|) = r\} = \{z \mid d_{\text{hyp.}}(z,a) = r\},$$

where $\theta = \text{Arg}(a)$. So

$$c_{\text{euc.}} = ce^{i\theta} = a\frac{1-\tanh^2(\frac{r}{2})}{1-|a|^2\tanh^2(\frac{r}{2})}$$

This also proves that the hyperbolic center and radius are unique.

Conversely, if we have a Euclidean circle centered at $a \in D \setminus \{0\}$ with radius $0 < r < 1-|a|$, we first consider the set

$$C := \{z \ ; \ \big|z - |a|\big| = r\}.$$

We set $x := |a| - r$ and $y := |a| + r$. We would like to find an $\alpha \in (0,1)$ such that if $T(z) := \frac{z-\alpha}{1-\alpha z}$ then $T(x) = -T(y)$. Using the quadratic formula, such an α exists and

$$0 < \alpha = \frac{xy + 1 - \sqrt{(1-x^2)(1-y^2)}}{x+y} < x < 1.$$

Now since $\overline{T(z)} = T(\overline{z})$ and T sends Euclidean circles to Euclidean circles, we have that the imaginary part of the center of $T(C)$ is 0. Since $T(x) = -T(y)$, we have that the real part of the center of $T(C)$ is 0, so the center of $T(C)$ is 0. The Euclidean radius of $T(C)$ is $T(x)$. From the above discussion, we have that if $r' = 2\,\text{arctanh}\, T(x)$ then

$$\begin{aligned} T(C) = \{z \mid d_{\text{hyp.}}(0,z) = r'\} \implies C &= \{T^{-1}(z) \mid d_{\text{hyp.}}(z,0) = r'\} \\ &= \big\{z \mid d_{\text{hyp.}}\big(T(z), T(\alpha)\big) = r'\big\} \\ &= \{z \mid d_{\text{hyp.}}(z,\alpha) = r'\}. \end{aligned}$$

Thus the hyperbolic center of C is α. Now if $\theta = \text{Arg}(a)$, we have

$$\begin{aligned} Ce^{i\theta} = \{z \ ; \ |z-a| = r\} &= \{ze^{i\theta} \mid d_{\text{hyp.}}(z,\alpha) = r'\} \\ &= \{z \mid d_{\text{hyp.}}(ze^{-i\theta},\alpha) = r'\} \\ &= \{z \mid d_{\text{hyp.}}(z,\alpha e^{i\theta}) = r'\}. \end{aligned}$$

Thus the center and radius of the hyperbolic circle are

$$c_{\text{hyp.}} = \alpha e^{i\theta} \quad \text{and} \quad r_{\text{hyp.}} = r' = 2\,\text{arctanh}\, T(x).$$

Definition 1.12. Given two distinct points on a general Riemannian manifold, the *geodesic* is a shortest path between them, parameterized by arc length. Corollary 1.8 not only gives the geodesic between two points in D, but it also motivates the definition of the geodesic between two points in $\overline{D}$. If $z_1 \in D$ and $z_2 \in \partial D$, then the geodesic between them is defined as $\varphi_{z_1,z_2}^{-1}([0,1])$. If $z_1, z_2 \in \partial D$, we know there exists a Möbius transformation φ such that $\varphi(z_1) = -1$ and $\varphi(z_2) = 1$. We can construct this φ as follows. If $z^* \in \partial D$ is distinct from $z_1, z_2, -1, 1$, we know there is an automorphism φ on the Riemann sphere such that $\varphi(z_1) = -1, \varphi(z_2) = 1$, and $\varphi(z^*) = z^*$. Since automorphisms of the Riemann sphere send circles to circles, $\varphi(\partial D) = \partial D$. Since φ is a homeomorphism, it sends connected components to connected components, so composing with an inversion if needed, we have our desired map. So we define the geodesic as $\varphi^{-1}([-1, 1])$. It can be seen that this definition is well-defined by geometric considerations, including the fact that right angles are preserved.

We next define the hyperbolic triangle.

Definition 1.13. Given three distinct points $z_1, z_2, z_3 \in \overline{D}$, the *hyperbolic triangle* with vertices z_1, z_2, z_3 is the closed region enclosed by the points and the geodesics connecting them. We can also describe hyperbolic triangles in terms of externally tangent hyperbolic circles. If $(r_1, r_2, r_3) \in (0, \infty]^3$, we say that a hyperbolic triangle with vertices z_1, z_2, z_3 is a *(r_1, r_2, r_3)-triangle* if there are hyperbolic circles $C_{r_1}(z_1), C_{r_2}(z_2), C_{r_3}(z_3)$ that are externally tangent to each other, where a circle of infinite radius is internally tangent to D with center on ∂D (we call these circles *horocycles*). Note that this implies that the side lengths opposite of z_1, z_2, z_3 have length $r_2 + r_3, r_1 + r_3, r_1 + r_2$ respectively. It can be shown that for any $(r_1, r_2, r_3) \in (0, \infty]^3$ there exists a unique (r_1, r_2, r_3)-triangle in $\overline{D}$ up to reflections and Möbius automorphisms (this is left as an exercise for the reader for now. One can use Möbius automorphisms to fix two of the hyperbolic circles, and then consider possible centers for the third circle). Consequently, there is a well-defined angle $\alpha_i(r_1, r_2, r_3) \in [0, \pi)$ subtended by the geodesics intersecting at vertex z_i for $i = 1, 2, 3$.

The next proposition is important in further understanding hyperbolic geometry.

Proposition 1.14 (Hyperbolic Law of Cosines)**.** *Let $(r_1, r_2, r_3) \in (0, \infty]^3$. Then if $r_1 < \infty$ we have*

$$\cos \alpha_1(r_1, r_2, r_3) = \begin{cases} \frac{\cosh(r_1+r_2)\cosh(r_1+r_3)-\cosh(r_2+r_3)}{\sinh(r_1+r_2)\sinh(r_1+r_3)} & \text{if } r_2, r_3 < \infty \\ \frac{\cosh(r_1+r_3)-e^{r_3-r_1}}{\sinh(r_1+r_3)} & \text{if } r_2 = \infty, r_3 < \infty \\ \frac{\cosh(r_1+r_2)-e^{r_2-r_1}}{\sinh(r_1+r_2)} & \text{if } r_2 < \infty, r_3 = \infty \\ 1 - 2e^{-2r_1} & \text{if } r_2 = r_3 = \infty \end{cases}.$$

Also, $\alpha_1(\infty, r_2, r_3) = 0$ for all $r_2, r_3 \in (0, \infty]$. In particular, $\alpha_1 : (0, \infty]^3 \to [0, \pi)$ is a continuous function (on the product topology of the subspace topology inherited by the extended real numbers).

Proof. Let $r_1 < \infty$. We will only deal with the case where $r_2, r_3 < \infty$ as the other cases follow from easier computations. Let z_1, z_2, z_3 be a (r_1, r_2, r_3)-triangle. The idea of this proof is to apply the Euclidean law of cosines to the Euclidean triangle connecting the Euclidean centers of these hyperbolic triangles. Since automorphisms of D are angle preserving, we can let $z_1 = 0$, $z_2 \in \mathbb{R}_{>0}$, and $z_3 \in D$. Note that

$$r_1 + r_2 = 2 \operatorname{arctanh} |z_2|, \quad \text{and} \quad r_1 + r_3 = 2 \operatorname{arctanh} |z_3|$$

by Corollary 1.8 and 1.9. Rearranging, we see

$$|z_2| = \tanh \frac{r_1 + r_2}{2}, \quad \text{and} \quad |z_3| = \tanh \frac{r_1 + r_3}{2} \tag{1.15}$$

Now let r_1', r_2', r_3' be the Euclidean radii of the hyperbolic circles $C(z_1, r_1), C(z_1, r_1)$ and $C(z_1, r_1)$ respectively, as given by the formula (1.11). In particular, we have

$$\begin{aligned}
r_1' &= \tanh(\tfrac{r_1}{2}) \\
r_2' &= \tanh(\tfrac{r_2}{2}) \cdot \frac{1 - |z_2|^2}{1 - |z_2|^2 \tanh^2(\frac{r_2}{2})} \overset{(1.15)}{=} \tanh(\tfrac{r_2}{2}) \cdot \frac{1 - \tanh^2 \frac{r_1+r_2}{2}}{1 - \tanh^2 \frac{r_1+r_2}{2} \tanh^2 \frac{r_2}{2}} \\
r_3' &= \tanh(\tfrac{r_3}{2}) \cdot \frac{1 - |z_3|^2}{1 - |z_3|^2 \tanh^2(\frac{r_3}{2})} \overset{(1.15)}{=} \tanh(\tfrac{r_3}{2}) \cdot \frac{1 - \tanh^2 \frac{r_1+r_3}{2}}{1 - \tanh^2 \frac{r_1+r_3}{2} \tanh^2 \frac{r_3}{2}}.
\end{aligned}$$

By the identity

$$\tanh y \cdot \frac{1 - \tanh^2(x+y)}{1 - \tanh^2(x+y)\tanh^2 y} = \frac{\sinh 2y}{\cosh 2y + \cosh(2x + 2y)},$$

for all x, y (which can be proven by factoring the fraction and converting to sinh and cosh), the formulas above then are

$$\begin{aligned}
r_1' &= \frac{\sinh \frac{r_1}{2}}{\cosh \frac{r_1}{2}} \\
r_2' &= \frac{\sinh r_2}{\cosh r_2 + \cosh(r_1 + r_2)} \\
r_3' &= \frac{\sinh r_3}{\cosh r_3 + \cosh(r_1 + r_3)}.
\end{aligned}$$

Now the Euclidean triangle joining the Euclidean centers of the hyperbolic circles. The Euclidean angle at $z_1 = 0$ is the same as $\alpha_1(r_1, r_2, r_3)$. This follows from the fact that the Euclidean and hyperbolic centers of a circle are on the same line through the origin (see Example 1.10), and hyperbolic geodesics emanating from the origin are lines. Thus by the Euclidean law of cosines, we have

$$\begin{aligned}
\cos \alpha_1(r_1, r_2, r_3) &= \frac{(r_1' + r_2')^2 + (r_1' + r_3')^2 - (r_2' + r_3')^2}{2(r_1' + r_2')(r_1' + r_3')} \\
&= 1 - 2\frac{r_2' r_3'}{(r_1' + r_2')(r_1' + r_3')}.
\end{aligned}$$

Substituting the values for r_1', r_2', r_3' yields the result.

Lastly, we have $\alpha_1(\infty, r_2, r_3)$ is between two geodesics making a right angle with ∂D from Corollary 1.8, thus $\alpha_1(\infty, r_2, r_3) = 0$. □

Definition 1.16. Let E be a Lebesgue measurable subset of D. Then the *hyperbolic area of E* is defined as the quantity

$$\text{area}_{\text{hyp.}}(E) := \int_E \frac{4 d\mu}{(1 - |z|^2)^2}.$$

It can by easily shown that hyperbolic area is invariant under Möbius transformations by the Schwartz-Pick lemma.

Proposition 1.17 (Hyperbolic Area Formulas)**.** *The area of a hyperbolic triangle is the difference of π and the sum of the triangle's interior hyperbolic angles. Additionally, the area of the interior of a hyperbolic circle with center p and radius $\rho < \infty$ is given by $4\pi \sinh^2(\rho/2)$.*

Proof. We will first verify the hyperbolic triangle area formula where one vertex is in D while the other two vertices are on ∂D. We first apply a Möbius transformation sending the interior vertex to the origin and one of the outer vertices to 1. If φ is the central hyperbolic angle formed, we finally rotate by $\varphi/2$ to get the picture in Figure 1. To find the Euclidean center and radius of the circle formed from the

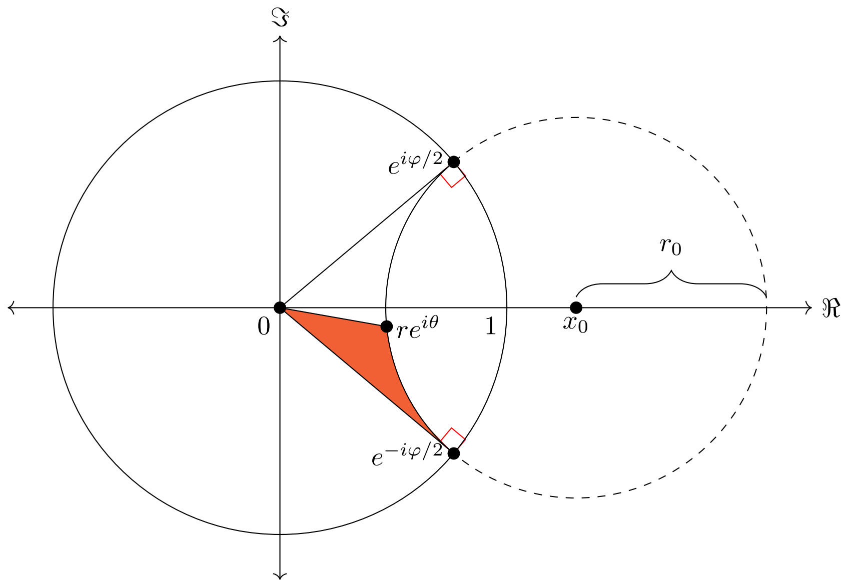

FIGURE 1. This figure illustrates the setup of the integration

geodesic, we implicitly differentiate $(x-x_0)^2+y^2=r_0^2$ to obtain

$$y' = \frac{x_0 - x}{y}.$$

Considering this at $e^{i\varphi/2}$, we set $(x,y) = (\cos\varphi/2, \sin\varphi/2)$, which implies $y' = \tan(\varphi/2)$, to deduce that $x_0 = \sec(\varphi/2)$ and $r_0 = \tan(\varphi/2)$. In order to integrate using polar coordinates, we need a formula for r in terms of θ. Using the formula of the circle, we solve for r in $\big(r\cos\theta - \sec(\varphi/2)\big)^2 + (r\sin\theta)^2 = \tan^2(\varphi/2)$ via the quadratic formula and obtain

$$r = f(\theta) = \frac{\cos\frac{\varphi}{2}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\frac{\varphi}{2}}}. \tag{1.18}$$

Setting up the integral, we see that the area of the triangle Δ is

$$\begin{aligned}
\int_\Delta \frac{4d\mu}{(1-|z|^2)^2} &= \int_{-\frac{\varphi}{2}}^{\frac{\varphi}{2}} \int_0^{f(\theta)} \frac{4r}{(1-r^2)^2} dr d\theta \\
&= \int_{-\frac{\varphi}{2}}^{\frac{\varphi}{2}} \left[\frac{2}{1-r^2}\right]_{r=0}^{r=f(\theta)} d\theta \\
&= \int_{-\frac{\varphi}{2}}^{\frac{\varphi}{2}} \frac{2d\theta}{1-f(\theta)^2} - 2\varphi
\end{aligned}$$

$$\overset{(1.18)}{=} \int_{-\frac{\varphi}{2}}^{\frac{\varphi}{2}} \left(\frac{\cos\theta}{\sqrt{\cos^2\theta - \cos^2\frac{\varphi}{2}}} + 1 \right) d\theta - 2\varphi$$

$$= \int_{-\frac{\varphi}{2}}^{\frac{\varphi}{2}} \frac{\cos\theta d\theta}{\sqrt{\sin^2\frac{\varphi}{2} - \sin^2\theta}} - \varphi$$

$$= \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} - \varphi = \pi - \varphi \tag{1.19}$$

where the penultimate equality follows from the u-substitution $u\sin(\varphi/2) = \sin\theta$. We thus proved the result in this case as the other two angles are 0 by Proposition 1.14. For triangles with one boundary vertex and two interior vertices, we can use Möbius transformation to map one of the interior vertices to 0 and the boundary vertex to 1, leaving an interior vertex z. We then extend the geodesic between 0 and z to the edge of the unit circle, intersecting it at z'. We then draw the unique geodesic connecting z' with 1 (see the Figure 2). To find the area of the triangle

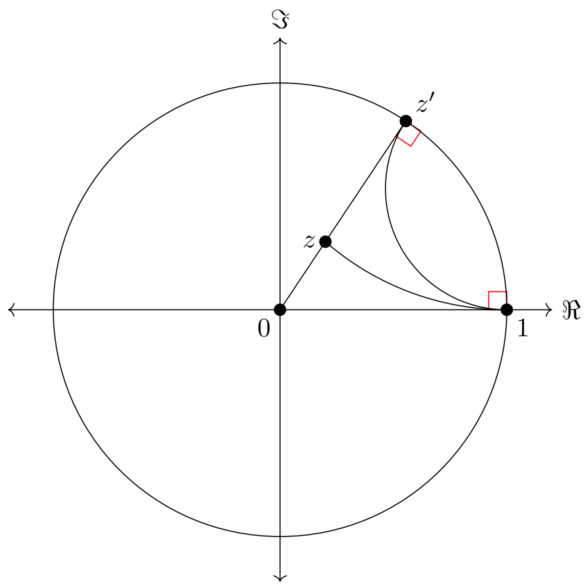

FIGURE 2. This figure shows the area trick

with vertices $0, 1, z$, it suffices to find the area of the triangle with vertices $0, 1, z'$ and subtract the area of the triangle with vertices $1, z, z'$, both of which can be computed with (1.19). A similar trick can be done when all vertices are interior. If all vertices are on the boundary, we apply a Möbius transformation mapping an interior point of the triangle to 0. From here, if we connect each boundary vertex to the origin via geodesics (they are lines since they contain the origin), we have split the boundary triangle into three triangles, each of which have two boundary vertices and one interior vertex. If we add the area formulas for these triangles, we get π, matching up with the formula in the proposition.

For the formula for a hyperbolic circle with center p and radius ρ, we can apply a Möbius transformation to map p to 0. Using a result deduced by Corollary 1.9, we see that $d_{\text{hyp.}}(0,z) = \rho$ if and only if $|z| = \tanh(\rho/2)$. So the area of circle C is given by

$$\begin{aligned}\int_C \frac{4d\mu}{(1-|z|^2)^2} &= \int_0^{2\pi}\int_0^{\tanh(\rho/2)} \frac{4r}{(1-r^2)^2} dr d\theta \\ &= 2\pi \left[\frac{2}{1-r^2}\right]_{r=0}^{\tanh(\rho/2)} \\ &= \frac{4\pi}{1-\tanh^2(\rho/2)} - 4\pi = 4\pi \sinh^2(\rho/2) \qquad \square\end{aligned}$$

We end this section with a statement of monotonicity.

Proposition 1.20. *The angle $\alpha_1(r_1, r_2, r_3)$ is strictly decreasing in r_1 and strictly increasing in r_2 and r_3 where the other two radii are fixed. Similarly, if $(r_1, r_2, r_3), (r_1', r_2', r_3') \in (0,\infty]^3$ such that $r_i \le r_i'$ for $i = 1, 2, 3$, then for a (r_1, r_2, r_3)-triangle T and a (r_1', r_2', r_3')-triangle T' we have*

$$\text{area}_{\text{hyp.}}(T) \le \text{area}_{\text{hyp.}}(T')$$

where equality holds if and only if $r_i = r_i'$ for $i = 1, 2, 3$.

One can prove this result by applying calculus to the formulas in Propositions 1.14 and 1.17 (for a geometric proof, see Lemma 2 on page 589 of [4]).

2. Circle Packings

In this section, we give a proof of the circle packing theorem stated below. We begin by defining the main object of this thesis: circle packings.

Definition 2.1. A (finite) *plane circle packing* is a (finite) collection $(C_i)_{i\in I}$ of circles $C_i = \{z \in \mathbb{C}\ ;\ |z - z_j| = r_j\}$ whose interiors are all disjoint and whose union is connected. The *nerve* of a circle packing is the graph whose vertices $\{z_i\ ;\ i \in I\}$ are the Euclidean centers of the circle packing, with two such centers connected by an edge if the circles are tangent.

Definition 2.2. A *hyperbolic circle packing* is a circle packing in $\overline{D}$ where all circles are hyperbolic circles.

It is worth noting that all graphs in this thesis are undirected and simple. It is clear by definition that the nerve of a circle packing is connected and planar (planar means it can be embedded into $\mathbb{C}$). An abstraction of the above Definition 2.1 is the concept of a Riemann sphere circle packing.

Definition 2.3. A *Riemann sphere circle* is a circle in $\mathbb{C}$ or a line in $\mathbb{C}$ (with ∞) together with one of the two connected components of the compliment, designated as the 'interior.' In the case of the circle in $\mathbb{C}$, if one chooses the unbounded region for the interior, the circle is called an *exterior circle*. A (finite) *Riemann sphere circle packing* is a (finite) collection $(C_i)_{i\in I}$ of Riemann sphere circle whose interiors are all disjoint and whose union is connected. The *nerve* of a Riemann circle packing is the graph whose vertices $\{z_i\ ;\ i \in I\}$ are the spherical centers of the Riemann circles, with two such centers connected by an edge if the circles are tangent.

Note that the nerve of a Riemann sphere circle packing is also planar, since we can apply a Möbius transformation to move all the points in the graph away from infinity. In light of the nerve being a graph, we next introduce maximality of connected planar graphs.

Proposition 2.4. *Let G be a connected planar graph with $n \geq 3$ vertices. Then the following are equivalent:*

(i) G is a maximal planar graph. That is, G is a planar graph such that no further edge can be added to G without making it either not simple or not planar.
(ii) G has $3n - 6$ edges.
(iii) Every drawing $\mathcal{D}$ of G divides the plane into faces that have three edges each (including the unbounded face).
(iv) At least one drawing $\mathcal{D}$ of G divides the plane into faces that have three edges each.

A discussion of this proposition may be found in Chapter 4 of [5]. By Corollary 4.4.7 in [5], every maximal planar graph with at least four vertices is 3-connected, which implies that each face has a unique boundary (see Lemma 4.2.5 of the same reference). In light of the above proposition and this fact, we define triangulation.

Definition 2.5. A *triangulation* $\mathcal{T}$ of a maximal planar graph G of at least four vertices is a drawing of G on the oriented plane, oriented Riemann sphere, or oriented hyperbolic disc, depending on context, with the faces included. An *oriented* *circle packing for a triangulation $\mathcal{T}$* is a circle packing (C_v) indexed by the vertices of $\mathcal{T}$ such that

(i) C_u and C_v are externally tangent if u and v are connected by an edge in $\mathcal{T}$.
(ii) C_u, C_v and C_w form a positively oriented triple whenever (u, v, w) forms a positively oriented face (the drawing of the nerve of (C_v) creates a well-defined face for each triple, and a positively oriented triple is respect to the orientation of the boundary of the corresponding oriented face).

If we neglect the second property, we simply call (C_v) a *circle packing for $\mathcal{T}$*.

The next theorem is the main theorem we would like to prove in this section.

Theorem 2.6 (Riemann Sphere Circle Packing Theorem)**.** *Let G be a maximal planar graph with at least four vertices, drawn as a triangulation $\mathcal{T}$ of the oriented Riemann sphere. Then there exists an oriented circle packing for $\mathcal{T}$, which is unique up to Möbius transformations.*

This theorem has the following important corollary

Corollary 2.7. *Let G be a connected planar graph with a drawing $\mathcal{D}$ on the plane. Then there exists a plane circle packing for $\mathcal{D}$. Furthermore, if G is maximal, this packing is unique up to reflections and Möbius transformations.*

Note that if the plane in this corollary is oriented and G is a maximal planar graph with at least four vertices with drawing $\mathcal{D}$, then there exists an oriented plane circle packing for $\mathcal{D}$ that unique up to Möbius transformations. So the reflection is there to reverse orientation, if needed.

Proof of Corollary 2.7. Note that the existence of a circle packing is easy to see when the connected planar graph has less than four vertices. The uniqueness statement is also easy to see in this case, as every packing of a maximal graph of one, two and three vertices is equal to $\{\Re z = 1\}$, $\{\Re z = \pm 1\}$, and $\{\Re z = \pm 1\} \cup \{|z| = 1\}$ respectively, up to Möbius transformation. Now if $\mathcal{D}$ has at least four vertices, if it is not maximal, we can add a vertex at the interior of each non-triangular face, and connect that vertex to the vertices of the face, to create a new maximal planar graph $\mathcal{D}'$. By Theorem 2.6, there exists a Riemann sphere circle packing for $\mathcal{D}'$. We use stereographic projection to project this Riemann sphere circle packing on to $\mathbb{C}$ (after a rotation to move ∞ away from the circle packing, if needed) to obtain a circle packing in $\mathbb{C}$ whose nerve is G'. After removing the circles corresponding to the added vertices, we have a circle packing whose nerve is G, as desired.

For the uniqueness statement, suppose $\mathcal{C}$ and $\mathcal{C}'$ are two circle packings for D, a drawing of a maximal planar graph with at least four vertices. After applying stereographic projection on $\mathcal{C}$ and $\mathcal{C}'$, then either both packings satisfy the orientation condition of Definition 2.5 for some orientation of the sphere, or they differ. If they differ, we can apply a reflection on one of the packings so they agree. Then they are equal up to Möbius transformation by Theorem 2.6. $\square$

In literature, the first statement of this corollary is called the *circle packing theorem* where is the second is called the *Koebe-Andreev-Thurston theorem*.

The idea of the proof of Theorem 2.6 can be split up into three main steps. The first step is to reduce Theorem 2.6 into an induction procedure that takes a 'weak' version of a hyperbolic circle packing of $D(0, 1)$ to a homotopic 'weak' hyperbolic circle packing of $\overline{D(0,1)}$ where all the boundary circles are horocycles. The second step is to prove that one can do this using a discrete version of Perron's method of solving the Dirichlet problem. The last step is to verify that the resulting 'weak' hyperbolic circle packing is indeed a genuine circle packing of $\overline{D(0,1)}$.

For the induction procedure, we need to introduce a technical graph theory concept.

Definition 2.8. Let G be a maximal planar graph with at least four vertices drawn as a triangulation $\mathcal{T}$. It can be seen that the degree of each vertex of $\mathcal{T}$ is at least three. Now starting from some arbitrary neighbor of a vertex v, we write the neighbors of v in order (with respect to the orientation) as $v_1, v_2, \ldots, v_d$. We see that v_i is adjacent to v_{i-1} and v_{i+1} (with $v_0 := v_d$ and $v_{d+1} := v_1$). We say v is *non-degenerate* if the following two conditions hold:

(i) There are no further adjacencies between the $v_1, \ldots, v_d$.
(ii) $\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$ is nonempty ($\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$ denotes $\mathcal{T}$ with the vertices $\{v, v_1, \ldots, v_d\}$ and all the emanating edges from these vertices deleted).

We call $v_1, v_2, \ldots, v_d$ the *boundary vertices* of $\mathcal{T} \setminus \{v\}$, and we call the other vertices the *interior vertices* of $\mathcal{T} \setminus \{v\}$.

This concept of non-degeneracy may be unmotivated, but the next proposition gives an equivalence, and this will be used in an inductive procedure.

Proposition 2.9. *Let G be a maximal planar graph with at least four vertices drawn as a triangulation $\mathcal{T}$. For $v \in \mathcal{T}$ with neighbors $v_1, \ldots, v_d$, the following are equivalent:*

(i) v is non-degenerate.
(ii) The graph $\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$ is nonempty and connected, and each vertex $v_1, \ldots, v_d$ is adjacent to at least one vertex in $\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$.

Proof. We start with the harder direction (i) $\implies$ (ii).

Step 1: We first show each vertex $v_1, \ldots, v_d$ is adjacent to at least one vertex in $\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$. We do this by the contrapositive statement. Assume there exists a neighbor v_i that is not adjacent to any vertex in $\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$. The triangular face bounded by the edge $v_i v_{i+1}$ that doesn't contain v must have $v_{i-1} v_i$ as another edge (since v_i has no other neighbors). Thus $v_{i-1} v_{i+1}$ is the last edge of this face, so v is degenerate.

Step 2: We next show $\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$ is connected. We first prove the following lemma.

Lemma 2.10. *Let $w, w' \in \mathcal{T} \setminus \{v, v_1, \ldots, v_n\}$ be vertices adjacent to v_i, v_{i+1} respectively for some i. Then w and w' are connected by a path in $\mathcal{T} \setminus \{v, v_1, \ldots, v_n\}$.*

Proof. Consider the triangular face bounded by the edge $v_i v_{i+1}$ that doesn't contain v. This face has a third vertex, which we call w^*. Note that w^* could be either w or w', but since v is non-degenerate, it cannot be any of $v, v_1, \ldots, v_d$. Thus $w^* \in \mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$. Next, consider all the edges emanating from v_i that are in the angle (not containing the edge $v_i v$) between the edges $v_i w$ and $v_i w^*$. Call the end of these edges $w_1, \ldots, w_n$ which are ordered from smallest to largest angle with $v_i w$. Again, we have $w_1, \ldots, w_n \in \mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$ since v is non-degenerate. By construction, $v_i w$ and $v_i w_1$ bound a triangular face, so w and w_1 are adjacent.

Similarly, we see w_j and w_{j+1} are adjacent, ending with w_n being adjacent to w^*. We have thus found a path in $\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$ connecting w to w^*. We can do the same argument to find a path in $\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$ connecting w^* and w'. Concatenating these paths yields the result. □

Now the connectedness of $\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$ follows easily from the lemma. Indeed, if $u, u' \in \mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$, we first connect u and u' with a path in $\mathcal{T}$. This path can be broken up into sub-paths, one path connecting u to some v_i (where all the vertices in the path are in $\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$ except v_i), another path connecting v_i to v_j in the subgraph $\{v, v_1, \ldots, v_d\}$, and the last path connecting v_j to u' (where all the vertices in the path are in $\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$ except v_j). If we let w be penultimate vertex in the first path and w' the second vertex in the third path, we can use Lemma 2.10 and Step 1 to create a path connecting w and w' in $\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$. Then u and u' are connected in $\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$ by following the sub-path connecting u to w, then the one created connecting w to w', and then finally the sub-path w' to u'.

Finally we show (ii) $\implies$ (i) by contrapositive. If v is degenerate, then there exists an additional adjacency between the neighbors $v_1, \ldots, v_d$ of v. By reordering the neighbors, we can assume v_1 is adjacent to v_k for some $3 \leq k < d$. Note that v_2 and v_d lie in different regions enclosed by the loop $v_1, v, v_3, \ldots, v_k, v_1$ (this is because the face triangles vv_1v_2 and vv_1d lie in different regions of this loop). It follows that either there exists a vertex (either v_2 or v_d) that is not adjacent to any vertex of $\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$, or $\mathcal{T} \setminus \{v, v_1, \ldots, v_d\}$ is not connected, as desired. □

We can now state the inductive procedure.

Theorem 2.11 (Inductive Step)**.** *Let G be a maximal planar graph with at least four vertices V with a triangulation $\mathcal{T}$ on the oriented Riemann sphere with v a non-degenerate vertex of $\mathcal{T}$ with neighbors $v_1, \ldots, v_d$. Suppose there exists an oriented Riemann sphere circle packing $\mathcal{C} := (C_w)_{w \in V \setminus \{v\}}$ for at least $\mathcal{T} \setminus \{v\}$ (meaning it satisfies a similar orientation condition, but we allow for extra tangencies between the circles, not just the ones required from the edges in the definition). Then there is an oriented Riemann sphere circle packing $\widetilde{\mathcal{C}} := (\widetilde{C}_w)_{w \in V}$ for $\mathcal{T}$. Furthermore, this packing is unique up to Möbius transformations.*

Proof of Theorem 2.6 using Theorem 2.11. We induct on the number of vertices of G. For the base case, let G be a maximal planar graph of four vertices, drawn as a triangulation $\mathcal{T}$ of the Riemann sphere. It is easy to see that G must be K^4, the graph of four vertices where every two vertices are connected by an edge. If $r := \frac{2\sqrt{3}}{3} - 1$, then

$$C(0, r),\ C(r+1, 1),\ C\big((r+1)e^{2\pi i/3}, 1\big),\ C\big((r+1)e^{4\pi i/3}, 1\big)$$

is a circle packing in $\mathbb{C}$ for $\mathcal{T}$ (where $C(z, \rho) := \{w \mid |w - z| = \rho\}$, see Figure 3). Viewing this circle packing as a Riemann sphere circle packing via stereographic projection, after assigning vertices of $\mathcal{T}$ to the circles so as to respect orientation, we have an oriented Riemann sphere circle packing for $\mathcal{T}$. For the uniqueness statement, given a circle packing, we can apply an inversion on a point of tangency and then use rotations and scaling to obtain $\{\Re z = \pm 1\} \cup \{|z \pm i| = 1\}$ (see Figure 3).

Now let G be a maximal planar graph with more than four vertices, drawn as a

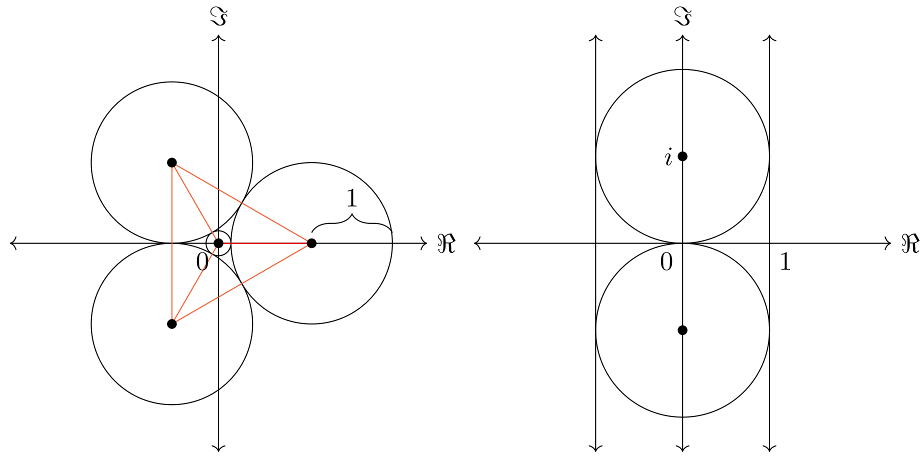

FIGURE 3. This figure shows the base case for Theorem 2.6

triangulation $\mathcal{T}$ on the Riemann sphere. We have two cases.

Case 1: Suppose $\mathcal{T}$ has a non-degenerate vertex v with neighbors $v_1, \dots, v_d$. We form another triangulation $\mathcal{T}'$ by contracting the edge connecting v and v_1, which removes one vertex and three edges. By the induction hypothesis, there exists an oriented Riemann sphere circle packing for $\mathcal{T}'$, and hence at least for $\mathcal{T} \setminus \{v\}$. By Theorem 2.11, there exists an oriented Riemann sphere circle packing for $\mathcal{T}$ that is unique up to Möbius transformations.

Case 2: Suppose $\mathcal{T}$ contains a degenerate vertex v. Let V be the set of vertices of $\mathcal{T}$. By assumption, there exists an additional adjacency between the neighbors $v_1, \dots, v_d$ of v. By reordering the neighbors, we can assume v_1 is adjacent to v_k for some $3 \le k < d$. Let V' denote the vertices of $V \setminus \{v_1, \dots, v_d\}$ in the region enclosed by the loop $v_1, \dots, v_k, v_1$ that doesn't contain v. Similarly, let V'' denote the vertices of $V \setminus \{v_1, \dots, v_d\}$ in the region enclosed by the loop $v_k, \dots, v_d, v_1, v_k$ that doesn't contain v. We have then partitioned the vertices into two parts:

$$V = \{v, v_1, \dots, v_d\} \sqcup V' \sqcup V''.$$

Let $\mathcal{T}'$ be the restriction of $\mathcal{T}$ to the vertices $\{v, v_1, \dots, v_k\} \sqcup V'$, and let $\mathcal{T}''$ be the restriction of $\mathcal{T}$ to the vertices $\{v, v_k, \dots, v_d, v_1\} \sqcup V''$ (see Figure 4 for an example). Note that both $\mathcal{T}'$ and $\mathcal{T}''$ are triangulations, so by the induction hypothesis, there exists oriented Riemann sphere circle packings $\mathcal{C}' = (C'_v)$ and $\mathcal{C}'' = (C''_v)$ for $\mathcal{T}'$ and $\mathcal{T}''$ respectively. Now we can apply the Möbius transformation to $\mathcal{C}''$ that sends the mutually tangent circles $C''_v, C''_{v_1}, C''_{v_k}$ to the mutually tangent circles corresponding to C'_v, C'_{v_1}, C'_{v_k} in $\mathcal{C}'$. After this transformation, due to orientation, $\mathcal{C}''$ will lie in one of the two connected components of the compliment on C'_v, C'_{v_1}, C'_{v_k} (with their interiors), and $\mathcal{C}'$ will lie in the other. We can then combine $\mathcal{C}'$ and $\mathcal{C}''$ to an oriented Riemann sphere circle packing $\mathcal{C}$ for $\mathcal{T}$. Also, since a Möbius transformation is uniquely determined by where it sends three distinct points, the uniqueness of this circle packing up to Möbius transformations follows from the uniqueness of $\mathcal{C}'$ and $\mathcal{C}''$. $\square$

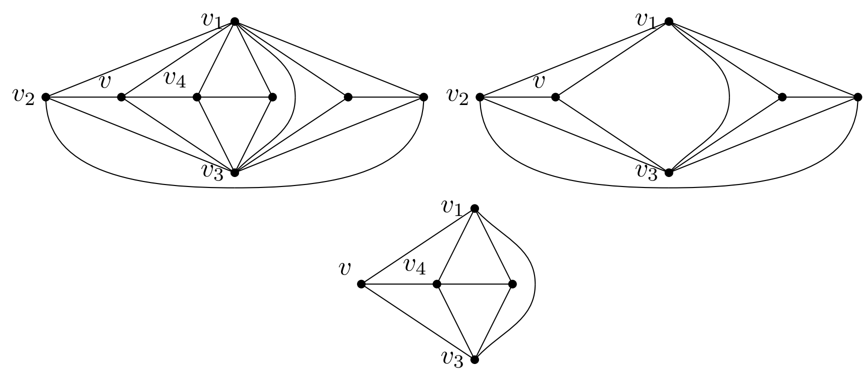

FIGURE 4. An example of a graph in the proof with $k = 3$ and $d = 4$ where $\mathcal{T}$ is in the upper left, $\mathcal{T}'$ is in the upper right, and $\mathcal{T}''$ in the middle

We reformulate Theorem 2.11 in the hyperbolic setting by the following theorem.

Theorem 2.12 (Hyperbolic formulation of the Inductive Step)**.** *Let G be a maximal planar graph with at least four vertices V with a triangulation $\mathcal{T}$ on the oriented hyperbolic disc with v a non-degenerate vertex of $\mathcal{T}$ with neighbors $v_1, \dots, v_d$. Suppose there exists an oriented hyperbolic circle packing $\mathcal{C} := (C_w)_{w \in V \setminus \{v\}}$ for at least $\mathcal{T} \setminus \{v\}$. Then there is an oriented hyperbolic circle packing $\widetilde{\mathcal{C}} := (\widetilde{C}_w)_{w \in V \setminus \{v\}}$ for $\mathcal{T} \setminus \{v\}$ where $C_{v_1}, \dots, C_{v_d}$ are horocycles. Furthermore, this packing is unique up to automorphisms of the unit disc.*

Proof of Theorem 2.11 using Theorem 2.12. Let $\mathcal{T}$ be a triangulation of a maximal planar graph G with at least four vertices V on the oriented Riemann sphere with a non-degenerate vertex v with neighbors $v_1, \dots, v_d$. If we pick a point p in one of the open faces of this triangulation, we can stereographically project this triangulation from p (and follow it with a scaling) to obtain a triangulation of the same graph on the hyperbolic disc (which we again call $\mathcal{T}$). Suppose there exists an oriented Riemann sphere circle packing $\mathcal{C} := (C_w)_{w \in V \setminus \{v\}}$ for at least $\mathcal{T} \setminus \{v\}$. We can again stereographically project from a point in the compliment of the circle packing (such that this point is not in any closure of any circle) to obtain an oriented hyperbolic circle packing (which we again call $\mathcal{C} := (C_w)_{w \in V \setminus \{v\}}$) for at least $\mathcal{T} \setminus \{v\}$. We can then apply Theorem 2.12 to get an oriented hyperbolic circle packing $\widetilde{\mathcal{C}} := (\widetilde{C}_w)_{w \in V \setminus \{v\}}$ for $\mathcal{T} \setminus \{v\}$ where $C_{v_1}, \dots, C_{v_d}$ are horocycles. We can stereographically project this up to the Riemann sphere to an oriented Riemann sphere circle packing. To this Riemann sphere circle packing, we add the equatorial circle C_v with center at the north pole. This new circle is externally tangent to $C_{v_1}, \dots, C_{v_d}$ since these circles were horocycles, and the addition of this new circle doesn't break the orientation condition of Definition 2.5 since it's neighboring triangles have their orientation preserved. The uniqueness statement of Theorem 2.11 follows directly from the uniqueness statement of Theorem 2.12 after rotating and scaling C_v to the equator with center at the north pole and stereographically projecting to the hyperbolic disc. $\square$

We have now reduced our problem to proving Theorem 2.12. The idea of the proof is to increase the radii of the circles so that the boundary circles are infinite. The problem is that one cannot do this arbitrarily without breaking the properties of circle packings. This is analogous to the fact that one cannot manipulate a harmonic function very easily. The Perron method works with subharmonic functions and takes a supremum to get a harmonic function. Analogously, we define a *subpacking* and take a supremum to get a genuine circle packing. Before we do that, we first give a remark.

Remark 2.13. The circle packing $\widetilde{\mathcal{C}} = (\widetilde{C})_{w\in V\setminus\{v\}} = \big(\widetilde{C}(\widetilde{p}_w, \widetilde{r}_w)\big)_{w\in V\setminus\{v\}}$ guaranteed by Theorem 2.12 is uniquely determined up to Möbius transformation by the values of the radii, $\widetilde{r}_w$. Indeed, given a triangle (t,u,w) in $\mathcal{T}\setminus\{v\}$, there is a (r_t, r_u, r_w)-hyperbolic triangle associated to this triangle formed by connecting the hyperbolic centers p_t, p_u, p_w of C_t, C_u, C_w in $\mathcal{C}$ via hyperbolic geodesics (we denote this $\Delta_{\mathcal{C}}(t,u,w)$). We can then form a $(\widetilde{r}_t, \widetilde{r}_u, \widetilde{r}_w)$-hyperbolic triangle (which we denote $\Delta_{\widetilde{\mathcal{C}}}(t,u,w)$) with the same orientation as $\Delta_{\mathcal{C}}(t,u,w)$, which is unique up to Möbius transformations by Definition 1.13 (since they have the same orientation). If we fix a triangle $\Delta_{\widetilde{\mathcal{C}}}(t,u,w)$, the adjacent triangles are also fixed by orientation, and we can continue this process to determine the entire circle packing.

Note that not all assignments of radii $\widetilde{r}_w$ have the desired properties of hyperbolic circle packings. We next introduce some weaker notions of hyperbolic circle packings with constraints we would like the radii to satisfy.

Definition 2.14.
- A *local packing* is an assignment of radii $r_w \in (0,\infty]$ to each vertex $w \in V\setminus\{v\}$ of $\mathcal{T}\setminus\{v\}$ in $\overline{D}$ that is under the following constraints:
  - Local constraint: If w is an interior vertex, the angles $\alpha_1(r_w, r_{w_1}, r_{w_2})$ around w sum to 2π.
  - Boundary constraint: The radii associated to the boundary vertices $v_1,\dots,v_d$ are infinite.
- A *local subpacking* is an assignment of radii $r_w \in (0,\infty]$ to each vertex $w \in V\setminus\{v\}$ of $\mathcal{T}\setminus\{v\}$ in $\overline{D}$ that is under the following weakened constraint:
  - Local sub-constraint: If w is an interior vertex, the angles $\alpha_1(r_w, r_{w_1}, r_{w_2})$ around w sum to *at least* 2π.

We begin with two lemmas about local subpackings.

Lemma 2.15 (Upper Bound)**.** *Let $(r_w)_{w\in V\setminus\{v\}}$ be a local subpacking. Then for any interior vertex w of degree d, one has $r_w \le \sqrt{d}$.*

Proof. Let $w \in V\setminus\{v\}$ be an arbitrary interior vertex. By the local sub-constraint of local subpackings, there is a hyperbolic (w,w_1,w_2)-triangle in V such that

$$\alpha_1(w,w_1,w_2) \ge \frac{2\pi}{d}.$$

Thus the (w,w_1,w_2)-triangle contains a sector of a hyperbolic circle of radius and angle $2\pi/d$. By 1.17, this triangle has area at most π, and thus we have

$$\pi \ge \frac{4\pi}{d}\sinh^2\left(\frac{r_w}{2}\right) \ge \frac{\pi r_w^2}{d}$$

where the last inequality we use the fact that $\sinh(x)\ge x$ for $x \ge 0$. We conclude $r_w \le \sqrt{d}$. ∎

Lemma 2.16 (Order). *If $\mathcal{R} = (r_w)_{w\in V\setminus\{v\}}$ and $\mathcal{R}' = (r'_w)_{w\in V\setminus\{v\}}$ are local subpackings, then $\max(\mathcal{R},\mathcal{R}') := \big(\max(r_w, r'_w)\big)_{w\in V\setminus\{v\}}$ is also a local subpacking.*

Proof. Let $w \in V \setminus \{v\}$ be an arbitrary interior vertex of $V \setminus \{v\}$. Without loss of generality, suppose $r_w \geq r'_w$. Now since $\mathcal{R}$ is a local subpacking, we have that angles $\alpha_1(r_w, r_{w_1}, r_{w_2})$ around w sum to at least 2π. Since $\alpha_1(r_w, r_{w_1}, r_{w_2})$ is strictly increasing in r_{w_1} and r_{w_2} by Proposition 1.20, we see that the angles

$$\alpha_1\big(\max(r_w, r'_w), \max(r_{w_1}, r'_{w_1}), \max(r_{w_2}, r'_{w_2})\big) = \alpha_1\big(r_w, \max(r_{w_1}, r'_{w_1}), \max(r_{w_2}, r'_{w_2})\big)$$

around w sum to at least 2π. □

We are now set to prove the following theorem.

Theorem 2.17 (Local Subpacking to Local Packing). *Suppose there exists a local subpacking on $V \setminus \{v\}$. Then there exists a unique local packing on $V \setminus \{v\}$.*

Proof. We first begin with proving existence. Define $\mathcal{R} = (r_w)_{w\in V\setminus\{v\}}$ to be the pointwise supremum of all the local subpackings on $V \setminus \{v\}$. By Lemma 2.15, $r_w < \infty$ for all interior vertices w. By Lemma 2.16, we can write $\mathcal{R}$ as a nondecreasing limit of local subpackings. In particular, $\mathcal{R}$ is also a local subpacking by the continuity of α_1 established in Proposition 1.14. Note that $\mathcal{R}$ satisfies the boundary constraint because if $r_{v_i} < \infty$ at some boundary vertex v_i, then by Proposition 1.20 we could replace r_{v_i} with ∞ and only increase the sum of the angles around the interior vertices connecting it while preserving the local-sub constraint of local subpackings, contradicting the maximality of $\mathcal{R}$. Finally, $\mathcal{R}$ satisfies the local constraint because if the sum of the angles $\alpha_1(r_w, r_{w_1}, r_{w_2})$ is strictly bigger than 2π at an interior vertex w, then by Propositions 1.14 and 1.20, we could increase r_w slightly while still maintaining the local-sub constraint at w and any other interior vertices, again contradicting the maximality of $\mathcal{R}$. Thus $\mathcal{R}$ is a local packing.
Finally, we establish uniqueness. Suppose $\mathcal{R}' = (r'_w)_{w\in V\setminus\{v\}}$ is a local packing. $\mathcal{R}'$ is in particular a subpacking, so by the maximality of $\mathcal{R}$, we have $r'_w \leq r_w$ for all $w \in V \setminus \{v\}$. By Proposition 1.20, we have

$$\begin{aligned} &\operatorname{area}(r'_w, r'_{w_1}, r'_{w_2}) \leq \operatorname{area}(r_w, r_{w_1}, r_{w_2}) \\ &\qquad\qquad \overset{\text{Prop.1.17}}{\iff} \sum_{j=1}^{3} \alpha_j(r'_w, r'_{w_1}, r'_{w_2}) \geq \sum_{j=1}^{3} \alpha_j(r_w, r_{w_1}, r_{w_2}) \end{aligned}$$

for any (w, w_1, w_2)-triangle. Summing over all triangles in $V \setminus \{v\}$, we have

$$\sum_{w\in V\setminus\{v\}} \sum_{(w,w_1,w_2)-\text{triangle}} \alpha_1(r'_w, r'_{w_1}, r'_{w_2}) \geq \sum_{w\in V\setminus\{v\}} \sum_{(w,w_1,w_2)-\text{triangle}} \alpha_1(r_w, r_{w_1}, r_{w_2}).$$

But by definition of local packings, the inner sum (on either side) is equal to 2π for an interior vertex and 0 for a boundary vertex. So the two sides agree, and by the equality statement in Proposition 1.20, we have $r_w = r'_w$ for all $w \in V \setminus \{v\}$, and we are done. □

We would now like to verify that this local packing is indeed a hyperbolic circle packing. This is a topological matter, and before we prove this, we introduce the notion of stars of vertices.

Definition 2.18. Let $\mathcal{R} = (r_w)_{w \in V \setminus \{v\}}$ be the unique local packing on $\mathcal{T} \setminus \{v\}$. If $w \in \mathcal{T} \setminus \{v\}$ has neighbors $w_1, \dots, w_d$, then by Definition 1.13, there exists a unique (r_w, r_{w_1}, r_{w_2})-triangle (which we call $\Delta_{\widetilde{\mathcal{C}}}(w, w_1, w_2)$) in $\overline{D(0,1)}$ with the same orientation as the triangle (w, w_1, w_2) in $\mathcal{T} \setminus \{v\}$ up to Möbius transformations. Once we fix $\Delta_{\widetilde{\mathcal{C}}}(w, w_1, w_2)$, the adjacent (r_w, r_{w_2}, r_{w_3})-triangle $\Delta_{\widetilde{\mathcal{C}}}(w, w_2, w_3)$ is also fixed. Continuing this way, we can define $\Delta_{\widetilde{\mathcal{C}}}(w, w_j, w_{j+1})$ for $j = 1, \dots, d$ where $w_{d+1} \coloneqq w_1$. These triangles are well-defined (i.e. it doesn't matter if we define them counterclockwise or clockwise around w) since the angles around w sum to 2π. We define the *star of w* to be the union of these hyperbolic triangles, and we denote it $\operatorname{star} w$. By Definition 1.13, $\operatorname{star} w$ is unique up to Möbius transformations, and the interiors of the triangles comprised in the star are disjoint by the local packing condition.

The idea of the following proof is to not place the circles, but we instead place the hyperbolic triangles by way of stars. We are essentially doing Remark 2.13 in reverse. This way, we can use familiar topological results in our favor. We first prove a topological lemma.

Lemma 2.19. *Let $f : X \to Y$ be a local homeomorphism between Hausdorff spaces with X compact and Y simply connected. Then f is a global homeomorphism.*

Proof. We wish to show f is a covering map. First we prove f is surjective. Since local homeomorphisms are open maps, $f(X)$ is open. Since f is also continuous, so $f(X)$ is compact. Since Y is Hausdorff, $f(X)$ is closed, and since Y is connected, $f(X) = Y$. To prove it is a covering map, let $y \in Y$. Since f is continuous, $f^{-1}(y)$ is closed, and since X is compact, $f^{-1}(y)$ is compact. Since f is a local homeomorphism, $f^{-1}(y)$ is discrete, so $f^{-1}(y) = \{x_1, \dots, x_n\}$ is finite. Let $U_1, \dots, U_n$ be mutually disjoint open neighborhoods of $x_1, \dots, x_n$ (which we can do since X is Hausdorff). By shrinking U_j if necessary, we may assume each U_j are mapped homeomorphically onto an open neighborhood V_j of y. If we let

$$K \coloneqq X \setminus (U_1 \sqcup \cdots \sqcup U_n) \text{ and } V \coloneqq (V_1 \cap \cdots \cap V_n) \setminus f(K)$$

we see K is closed (and hence compact implying $f(K)$ is compact and hence closed) and V is an open neighborhood of y. Then V is evenly covered since

$$f^{-1}(V) = \big(U_1 \cap f^{-1}(V)\big) \sqcup \cdots \sqcup \big(U_n \cap f^{-1}(V)\big).$$

Thus f is a covering map. Since V is simply connected, it is it's own universal cover, so there exists a covering map $f^{-1} : Y \to X$ such that $f \circ f^{-1} = \mathrm{id}_Y$ (i.e. the lift of $\mathrm{id}_Y : Y \to Y$). This equality implies f^{-1} is injective, and it is already surjective continuous open map as a covering map. So f^{-1} is a homeomorphism, implying f is a homeomorphism. $\square$

Theorem 2.20 (Local Packing to Circle Packing)**.** *Suppose $\mathcal{R} = (\widetilde{r}_w)_{w \in V \setminus \{v\}}$ is a local packing on $\mathcal{T} \setminus \{v\}$. Then there is a hyperbolic circle packing $\widetilde{\mathcal{C}} \coloneqq \big(\widetilde{C}(\widetilde{p}_w, \widetilde{r}_w)\big)_{w \in V \setminus \{v\}}$ for $\mathcal{T} \setminus \{v\}$.*

Proof. For every $w \in \mathcal{T} \setminus \{v\}$, we fix $\operatorname{star} w$ in $\overline{D(0,1)}$. Now let

$$M \coloneqq \coprod_{w \in \mathcal{T} \setminus \{v\}} \operatorname{star} w \Big/ \sim$$

where $z \in \operatorname{star} w \sim z' \in \operatorname{star} w'$ if w and w' are adjacent and $\psi_{w,w'}(z) = z'$ where $\psi_{w,w'}$ is the unique Möbius transformation taking $w, w' \in \operatorname{star} w$ to $w, w' \in \operatorname{star} w'$, respectively. Note that M is homeomorphic to $\mathcal{T} \setminus \{v\}$, so M is Hausdorff and simply connected. We will use these facts to show that the following map is well-defined. If we fix $w_0 \in \mathcal{T} \setminus \{v\}$, we let $\Psi : M \to \overline{D(0,1)}$ be defined as follows: Define $\Psi([z]) \coloneqq z$ for $z \in \operatorname{star} w_0$. For $z \in M$, we let $\gamma : [0,1] \to M$ be a path such that $\gamma(0) = w_0$ and $\gamma(1) = z$. We can partition this path into sub-paths $\gamma_j \subseteq \operatorname{star} w_j \subseteq M$ where $\gamma = \gamma_0 \cup \cdots \cup \gamma_k$. We then define auxiliary functions

$$\begin{aligned} \psi_0([w]) &\coloneqq w \text{ for } w \in \operatorname{star} w_0 \\ \psi_1([w]) &\coloneqq \psi_{w_1,w_0}(w) \text{ for } w \in \operatorname{star} w_1 \\ &\vdots \\ \psi_k([w]) &\coloneqq \psi_{w_k,w_{k-1}}(w) \text{ for } w \in \operatorname{star} w_k \end{aligned}$$

where ψ_{w_1,w_0} is the unique Möbius transformation taking $w_1, w_0 \in \operatorname{star} w_1$ to $w_1, w_0 \in \operatorname{star} w_0$, respectively, and ψ_{w_{j+1},w_j} is the unique Möbius transformation taking $w_{j+1}, w_j \in \operatorname{star} w_{j+1}$ to $w_{j+1}, w_j \in \psi_{w_j,w_{j-1}}(\operatorname{star} w_j)$, respectively. Finally, we define $\Psi([z]) \coloneqq \psi_k([z])$. By the general monodromy theorem applied to the étale space of the sheaf of holomorphic functions on M, we see that the values of Ψ on homotopic curves coincide (see [6] for the general monodromy theorem, and see [7] for the étale space). But since M is simply connected, all closed loops are null homotopic, so Ψ is well-defined. In particular, the stars of the vertices in the new packing are well-defined, and hence the hyperbolic triangles are well-defined.

To finish this argument off, we need to show the hyperbolic triangles have disjoint interiors. Note that the image of Ψ does not quite cover the closed disc. This is due to the boundary triangles' boundary edge not being the same as the respective arc on the unit circle, leaving a lens shaped region. We can modify Ψ to a map $\widetilde{\Psi}$ which deforms this boundary edge to the arc of the unit circle and leaves the other edges unchanged. This map $\widetilde{\Psi} : M \to \overline{D(0,1)}$ is a local homeomorphism. By Lemma 2.19, we see $\widetilde{\Psi}$ is a global homeomorphism, and so the original map Ψ is injective. Thus the hyperbolic triangles, along with the lens-shaped regions on the boundary, have disjoint interiors. Because of this, the circles associated to these hyperbolic triangles will all have disjoint interiors, proving the desired result. $\square$

3. Quasiconformal Maps

We begin this section with the theorem of central importance for which we give two proofs (the proofs are based on pages 224, 258 of [8]).

Theorem 3.1 (Riemann Mapping Theorem)**.** *Let $U \subsetneq \mathbb{C}$ be a non-empty, simply connected open set. If $z_0 \in U$, then there exists a unique biholomorphic map $f : U \to D$ such that*

$$f(z_0) = 0 \text{ and } f'(z_0) > 0.$$

Proof of Uniqueness. Suppose f and g are two such maps. Define $h : D \to D$ with $h(z) = f \circ g^{-1}(z)$. Then we have the following

- h and h^{-1} are biholomorphic
- $h(0) = h^{-1}(0) = 0$
- $h'(0) = f'\big(g^{-1}(0)\big) \cdot (g^{-1})'(0) = f'\big(g^{-1}(0)\big)/g'\big(g^{-1}(0)\big) > 0.$

Thus by Schwarz's lemma, $|h(z)| \le |z|$ and $|h^{-1}(w)| \le |w|$ for all $z, w \in D$. Letting $w = h(z)$, we see that $|h(z)| = |z|$ for all $z \in D$. By equality statement in Schwarz's lemma, we have that $h(z) = az$ where $|a| = 1$. Since $h'(0) > 0$, we deduce that $a = 1$ and thus $f \equiv g$. $\square$

Proof of Existence. Step 1: Let $\alpha \notin U$. Then $z - \alpha$ is non-zero on U, and hence, $\ell(z) := \log(z-\alpha)$ is well-defined on U. Since $e^{\ell(z)} = z-\alpha$, we see that ℓ is injective. Now if we let $s > 0$ be such that $B_s(z_0) \subseteq U$, then

$$\ell\big(B_s(z_0)\big) + 2\pi i \cap \ell(U) = \varnothing$$

for otherwise we would have $\ell(w) + 2\pi i = \ell(z)$ for some $w \in B_s(z_0)$ and $z \in U$. Exponentiating, we find $z = w$ and thus $\ell(z) = \ell(w)$ which is a contradiction. Furthermore, there exists $r > 0$ such that

$$B_r\big(\ell(z_0) + 2\pi i\big) \cap \ell(U) = \varnothing. \tag{3.2}$$

This is because $\ell\big(B_s(z_0)\big)+2\pi i$ is open (since ℓ is an open map by the open mapping theorem) so there exists $r > 0$ such that $B_r\big(\ell(z_0) + 2\pi i\big) \subseteq \ell\big(B_s(z_0)\big) + 2\pi i$. Now let $F : U \to \mathbb{C}$ be such that $F(z) := (T \circ I \circ \ell)(z)$ where

$$I(z) := \frac{1}{z - \big(\ell(z_0) + 2\pi i\big)} \quad \text{and} \quad T(z) = z + \frac{1}{2\pi i}.$$

Note that the inversion I is bounded by $1/r$ on $\ell(U)$ by (3.2). Furthermore, F is bounded by $M := \frac{1}{r} + \frac{1}{2\pi}$ on U by the triangle inequality. Now define $G := \frac{1}{2M}F$. Since ℓ is injective, so is G. Furthermore, $G(z_0) = 0$ and $|G(z)| < 1$.

Step 2: Without loss of generality, by composing with the map described in step 1, we may assume $U \subset D$ with $0 \in U$ (such a domain is called a *Koebe domain*). Define

$$\mathcal{F} := \{f : U \to D \text{ holomorphic, injective, and } f(0) = 0\}.$$

Note $\mathcal{F} \neq \varnothing$ because it contains the identity. Also, $\mathcal{F}$ is uniformly bounded because $|f(z)| < 1$ for all $f \in \mathcal{F}$ and $z \in U$. We want to find a function $f \in \mathcal{F}$ that maximizes $|f'(0)|$. The reason for this is we would like to maximize the 'spread' of values in the range so as to cover the whole disc. Define

$$s := \sup_{f \in \mathcal{F}} |f'(0)|.$$

Now there exists a sequence of functions $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ such that $|f_n'(0)| \xrightarrow{n\to\infty} s$. By Montel's theorem, there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ that converges uniformly to a function f on compact subsets of U. First, f is holomorphic on U by Weierstrass' theorem. Since $s \geq 1$ (because the identity is in $\mathcal{F}$), f is non-constant, hence injective by a corollary of Hurwitz's theorem. Also, $f(0) = 0$ and, by continuity, we have $|f(z)| \leq 1$ on U. Furthermore, by the maximum modulus principle, we see that $|f(z)| < 1$ and hence $f \in \mathcal{F}$ with $|f'(0)| = s$.

Step 3: We want to show f from step 2 is surjective. If this were not true, we could construct a function $g \in \mathcal{F}$ with $|g'(0)| > s$. Indeed, suppose there exists $\alpha \in D$ such that $f(z) \neq \alpha$ for all $z \in U$. Consider the automorphism of the unit disk

$$\psi_\alpha(z) := \frac{\alpha - z}{1 - \overline{\alpha} z}$$

that interchanges 0 and α. Since U is simply connected, so is $f(U)$. With ψ_α is nonzero on $f(U)$, there exists $h : f(U) \to D$ such that

$$h^2 = \psi_\alpha.$$

Now define

$$g := \psi_{h(0)} \circ h \circ f.$$

We claim $g \in \mathcal{F}$. It is easy to see g is holomorphic and $g(0) = 0$. It is also clear that g is injective, as each function in the composition is injective. Finally, by a computation, we see $|f'(0)| < |g'(0)|$. This is a contradiction with the definition of f and thus f is surjective. If $f'(0) = r_0 e^{i\theta_0} \neq 0$, define $\widetilde{f} : U \to D$ by

$$\widetilde{f}(z) := e^{-i\theta_0} f(z).$$

This function satisfies the condition $\widetilde{f}'(0) > 0$ and thus satisfies the theorem. □

In Step 3, the use of the square root function is perplexing at first. We would like the derivative to increase on U, so it is natural to consider the logarithm as it's derivative is greater than 1 on the unit disc. Doing the computations, we see that we would like a function on $\psi_\alpha\big(f(U)\big)$ that has the *reverse* inequality in the derivative inequality in the Schwartz-Pick lemma. The square root happens to have this property.

Another interpretation is that $\psi_{h(0)} \circ h$ is injective on $f(U)$, but it's inverse is not injective on D. So applying Schwartz lemma on $\Phi := (\psi_{h(0)} \circ h)^{-1}$, we see $|\Phi'(0)| < 1$, which implies $|g'(0)| > |\Phi'(0)||g'(0)| = |f'(0)|$. Another proof of the Riemann mapping theorem that expresses the desired map as a limit of functions is in the appendix.

This next theorem extends the Riemann mapping theorem and motivates the definition of quasiconformal maps (proof is based on page 110 of [9]).

Theorem 3.3 (Carathéodory's Theorem)**.** *Let $U \subseteq \mathbb{C}$ be non-empty, bounded, open, and simply connected, and let $f : D \to U$ be a biholomorphic map (shown to exist by Theorem 3.1). Then f extends to a continuous homeomorphism from $\overline{D}$ to $\overline{U}$ if and only if ∂U is a Jordan curve.*

Proof. $\underline{\Rightarrow}$: This is easy to see as homeomorphisms send boundary to boundary and Jordan curves to Jordan curves.
$\underline{\Leftarrow}$: To prove the continuous extension to the boundary, we will first prove for any

$\zeta \in \partial D$ we have that the diameters of the sets $f\big(D \cap B_{r_n}(\zeta)\big)$ converges to 0 for some sequence of radii $r_n \to 0$. If we prove this, we define the extension $\widetilde{f}$ for $\zeta \in \partial D$ as

$$\widetilde{f}(\zeta) := \bigcap_{n \geq 1} \overline{f\big(D \cap B_{r_n}(\zeta)\big)} \tag{3.4}$$

with $\widetilde{f}|_D = f$. This extension is sequentially continuous and hence continuous. This is easy to see for a sequence in D converging to a point on ∂D, and for a sequence in ∂D converging to a point on ∂D, we can apply a 3ε argument. These two statements give sequential continuity. To prove the diameters of the sets $f\big(D \cap B_{r_n}(\zeta)\big)$ converges to 0, we first consider areas. Note that U is Borel measurable with finite area (since it is open and bounded), and by a change of variables, we see that

$$\text{area}(U) = \int_D |f'(x)|^2 dxdy < \infty.$$

By considering polar coordinates around $\zeta \in \partial D$, we see that this integral is equal to

$$\int_0^2 \Bigg(\underbrace{\int_0^{2\pi} \mathbf{1}_D(\zeta + re^{i\theta}) |f'(\zeta + re^{i\theta})|^2 d\theta}_{:= g(r)} \Bigg) r dr < \infty$$

Note that for any $\varepsilon, \varepsilon' > 0$ there exists $x \in (0, \min\{\varepsilon', 2\})$ such that $x^2 g(x) < \varepsilon$ because, otherwise, we would contradict the divergence of the integral $\int_0^2 \frac{dr}{r}$ near $r = 0$. Thus there exists a strictly decreasing sequence $0 < r_n < 2$ converging to 0 such that $r_n^2 g(r_n) < 1/n$. Expanding this out, we see

$$\int_0^{2\pi} r_n^2 \mathbf{1}_D(\zeta + r_n e^{i\theta})^2 |f'(\zeta + r_n e^{i\theta})|^2 d\theta < \frac{1}{n}.$$

Hence, by Cauchy-Schwarz, we have

$$\int_0^{2\pi} r_n \mathbf{1}_D(\zeta + r_n e^{i\theta}) |f'(\zeta + r_n e^{i\theta})| d\theta < \frac{2\pi}{n}.$$

If we define $\ell_n := \{\zeta + r_n e^{i\theta} : 0 \leq \theta < 2\pi\} \cap D = \{\zeta + r_n e^{i\theta} : \alpha_n < \theta < \beta_n\}$, we see that $f(\ell_n)$ is a rectifiable curve in U with length equal to the integral above. In particular, the length of $f(\ell_n)$ goes to 0 as $n \to \infty$. Unfortunately, an arc on ∂U connecting the endpoints of $f(\ell_n)$ need not be rectifiable, so we need to find a bound for its diameter instead of its length. Since $f(\ell_n)$ has finite length, the endpoints of this curve, defined as a_n and b_n, are well-defined points. To see this, we use the Cauchy condition for limits of functions. If we assume this Cauchy condition doesn't hold, then for any given number, we can find a partition P for which the variation of $f(\ell_n)$ with respect to P exceeds this number, implying the curve is not of finite length (another way to see this Cauchy condition holds is with the absolute continuity of integrals). Because f is a proper map, we see that these endpoints are in ∂U. By definition of rectifiable, we have $|a_n - b_n| < 2\pi/n$ and hence the distance between a_n and b_n goes to 0 as $n \to \infty$. We use the next claim to bound the diameter of a certain arc connecting the endpoints of $f(\ell_n)$.

Lemma 3.5. *Let γ be a Jordan curve. Then there is a function $\eta(\delta)$, defined for all sufficiently small $\delta > 0$, with $\eta(\delta) \to 0$ as $\delta \searrow 0$, such that if $a, b \in \gamma$ with*

$|a-b|<\delta$ then there is an arc of γ having endpoints a, b whose diameter is less than $\eta(\delta)$ (we allow the single-point arc).

Proof. Denote the map $\gamma : S^1 \to \mathbb{C}$ the Jordan curve. Since γ is a bijective map of compact Hausdorff spaces, it has a uniformly continuous inverse. We let $\delta_0 > 0$ be small enough such that $|\gamma(\zeta)-\gamma(\zeta')|<\delta_0$ implies $|\zeta-\zeta'|<2$ where $\zeta,\zeta'\in S^1$. If $\zeta,\zeta'\in S^1$ with $|\gamma(\zeta)-\gamma(\zeta')|<\delta_0$, we let $\sigma(\zeta,\zeta')$ be the unique shorter arc of S^1 having endpoints ζ,ζ'. We claim

$$\operatorname{diam}\gamma|_{\sigma(\zeta,\zeta')}\to 0 \text{ uniformly as } |\gamma(\zeta)-\gamma(\zeta')|\to 0.$$

Indeed, by uniform continuity of γ, if $\varepsilon>0$, there exists $\delta'>0$ such that $|\gamma(\xi)-\gamma(\xi')|<\varepsilon$ whenever $|\xi-\xi'|<\delta'$. By uniform continuity of γ^{-1}, there exists $\delta''>0$ such that $|\zeta-\zeta'|<\delta'$ whenever $|\gamma(\zeta)-\gamma(\zeta')|<\delta''$. Now if $|\gamma(\zeta)-\gamma(\zeta')|<\min\{\delta'',\delta_0\}$, we have $|\zeta-\zeta'|<\min\{\delta',2\}$. Then clearly $|\xi-\xi'|<\delta'$ for all $\xi,\xi'\in\sigma(\zeta,\zeta')$, which implies $|\gamma(\xi)-\gamma(\xi')|<\varepsilon$ for all $\xi,\xi'\in\sigma(\zeta,\zeta')$, as desired.

If $0<\delta\le\delta_0$, then we set

$$\eta(\delta) := \sup\{\operatorname{diam}\gamma|_{\sigma(\zeta,\zeta')} : |\gamma(\zeta)-\gamma(\zeta')|<\delta\}.$$

We then have the desired result as $\eta(\delta)\to 0$ as $\delta\searrow 0$. □

Now we are set to prove $\operatorname{diam} f\big(D\cap B_{r_n}(\zeta)\big)\to 0$ as $n\to\infty$. If we fix $\varepsilon>0$, let N be large enough so that for all $n>N$ we have $r_n<1/2$ and $1/n$ is in the domain of η in Lemma 3.5 (applied to a_n and b_n) with

$$\pi\big(l_n+\eta(1/n)\big)^2 < A \quad\text{and}\quad l_n+\eta(1/n)<\varepsilon \tag{3.6}$$

where l_n is the length of $f(\ell_n)$ and A is the (positive) area of the image of f on the ball of radius $1/2$ centered at 0. Let γ_n be the arc guaranteed by Lemma 3.5, and let R_n be the bounded region in D with boundary $\gamma_n\cup f(\ell_n)$. Since f sends connected components to connected components, either $f\big(D\cap B_{r_n}(\zeta)\big)=R_n$ or $f\big(D\cap \overline{B_{r_n}(\zeta)}^c\big)=R_n$. Note that the area of R_n is bounded by $\pi\big(l_n+\eta(1/n)\big)^2$ since R_n is contained in the closed disc of radius $l_n+\eta(1/n)$ centered at a_n, where we use the fact that $\operatorname{diam} f(\ell_n)\le l_n$. Since $B_{1/2}(0)\subseteq D\cap\overline{B_{r_n}(\zeta)}^c$ and the area of $f\big(B_{1/2}(0)\big)$ is positive, we must have $f\big(D\cap B_{r_n}(\zeta)\big)=R_n$. Finally, since

$$\operatorname{diam} R_n=\operatorname{diam}\partial R_n\le l_n+\eta(1/n)\overset{(3.6)}{<}\varepsilon,$$

we have proved $\operatorname{diam} f\big(D\cap B_{r_n}(\zeta)\big)\to 0$ as $n\to\infty$.

We now prove that the extension $\widetilde{f}$ (defined at (3.4)) is bijective. By definition of $\widetilde{f}$ and the properness of f, we see $\widetilde{f}(\partial D)\subseteq\partial U$. Since $\widetilde{f}$ is proper (since it is continuous on the compact set $\overline{D}$), we see that $\widetilde{f}(\overline{D})$ is a compact subset of $\overline{U}$ containing U, so we see $\widetilde{f}$ is surjective. For $\widetilde{f}$ not to be injective, there would be distinct $\zeta,\zeta'\in\partial D$ such that $\widetilde{f}(\zeta)=\widetilde{f}(\zeta')$. If L is the line connecting ζ and ζ' in D, we see that $\widetilde{f}(L)$ is a Jordan curve in U only intersecting ∂U at $\widetilde{f}(\zeta)=\widetilde{f}(\zeta')$. Note f maps one of the connected components separated by L in D to the interior of the Jordan curve, and we deduce that $\widetilde{f}$ is constant on a non-degenerate arc of D. Composing with the Cayley transformation and applying the Schwarz reflection principle, we see that f is a constant by uniqueness of analytic continuation. But this contradicts the conformality of f. Thus the proof of injectivity is complete. □

We next define some notions involved with Jordan quadrilaterals.

Definitions 3.7.
- A *Jordan quadrilateral* is an open region $Q \in \mathbb{C}$ enclosed by a Jordan curve with four distinct points p_1, p_2, p_3, p_4 called the *vertices* of the quadrilateral (these points are in counterclockwise order).
- The *a-sides* are the arcs in ∂Q connecting p_1 to p_2 and p_3 to p_4, while the *b-sides* are the arcs in ∂Q connecting p_2 to p_3 and p_4 to p_1.
- A *vertex-preserving conformal map* from one Jordan quadrilateral Q to another Q' is a conformal map that extends to a homeomorphism from $\overline{Q}$ to $\overline{Q'}$ that maps the corners of Q to the respective corners of Q', and hence maps a-sides to a-sides and b-sides to b-sides.

The next proposition gives regularity to these definitions.

Theorem 3.8. *For every Jordan quadrilateral Q there exists a vertex-preserving conformal map $\psi : Q \to R$ where R is a Euclidean rectangle. Furthermore, this rectangle is unique up to complex affine transformations.*

Proof of Existence. By Theorems 3.1 and 3.3, there exists a conformal map $f : Q \to D$ that extends to a homeomorphism $\widetilde{f} : \overline{Q} \to \overline{D}$. If p_1, p_2, p_3, p_4 are the vertices of Q counterclockwise order, their image points under $\widetilde{f}$ are also in counterclockwise order by the orientation preservation of conformal maps. By composing $\widetilde{f}$ with a Möbius transformation of the disk, we may assume p_1 gets mapped to i, p_2 gets mapped to -1, and p_4 gets mapped to $-i$. Since $\widetilde{f}$ preserves the counter-clockwise orientation, p_3 gets mapped to $e^{i\theta_0}$ for some $\theta_0 \in (\pi, 3\pi/2)$. Note that the Möbius transformations of the unit disc that fix $-i$ and i are exactly

$$z \mapsto \frac{z - ix}{1 + ixz}, \quad x \in (-1, 1).$$

We claim that there exists an $x \in (-1, 1)$ such that the image of -1 and $e^{i\theta_0}$ under the above map are conjugate. Indeed, this is equivalent to finding a solution to

$$x^2 - 2\frac{\sin\theta_0}{1 + \cos\theta_0}x + 1 = 0$$

in $(-1, 1)$. This quadratic has two distinct real solutions as seen from the discriminant, and the product of these solutions is 1. Thus one solution must be in $(-1, 1)$ as desired. After composing $\widetilde{f}$ with this transformation, we may assume p_1, p_4 get mapped to $i, -i$ respectively, with v_2 and v_3 mapped to conjugates with arguments in $(\pi/2, 3\pi/2)$. We then compose with the inverse Cayley transformation,

$$g(z) = \frac{z + 1}{i(z - 1)},$$

we see that p_1, p_2, p_3, p_4 are mapped to $-1, -r, r, 1$ respectively for some $r \in (0, 1)$. From Example 3 on page 233 of [8], we see that the Schwarz-Christoffel elliptic integral

$$S(z) = \int_0^{\frac{z}{r}} \frac{d\zeta}{[(1 - \zeta^2)(1 - r^2\zeta^2)]^{1/2}}$$

maps the upper half-plane to a Euclidean rectangle R, mapping the points $-1, -r, r, 1$ to the vertices of R in a counterclockwise orientation, as desired. $\square$

Proof of Uniqueness. If $\psi : Q \to R$ and $\phi : Q \to R'$ are two such maps, we see that $f := \phi \circ \psi^{-1}$ is a vertex-preserving conformal map from R to R'. By composing with complex affine transformations, we may assume $R = (0, r) \times (0, 1)$

and $R' = (0, r') \times (0, 1)$ where $r, r' > 0$. Without loss of generality, we may assume $r' > r$. If we consider the four congruent boundary rectangles to R, we see that f can be extended to these rectangles by the Schwarz reflection principle. Continuing this way, we extend f to an entire function, which we still call f. By construction, we have

$$z \in \big[ri, r(i+1)\big] \times [j, j+1] \implies f(z) \in \big[r'i, r'(i+1)\big] \times [j, j+1]$$

for all $i, j \in \mathbb{Z}$. Hence for any $R > 0$ we have

$$\sup_{|z|=R} |f(z)| \leq \sup_{z \in [-R,R]^2} |f(z)| \leq \sup_{z \in \left[-\frac{Rr'}{r}, \frac{Rr'}{r}\right]^2} |z| = \frac{Rr'}{r}\sqrt{2}.$$

By Cauchy's differentiation formula, we see $f'' = 0$ and hence f is a complex affine transformation. $\square$

This proposition motivates the following definition.

Definition 3.9. The *conformal modulus* $\mathrm{mod}(Q)$ (or *modulus* for short) of a Jordan quadrilateral with vertices p_1, p_2, p_3, p_4 is the ratio $b/a > 0$, where a, b are the lengths of the a-sides and b-sides respectively of a rectangle R that is conformal to Q in a vertex-preserving fashion.

Note that Theorem 3.8 makes this well-defined, as well as show that $\mathrm{mod}(Q)$ is unchanged by vertex-preserving conformal maps. Also note that each cyclic permutation of the vertices replaces the modulus with its reciprocal. This observation will be key to understanding the next definition, the most important of this section.

Definition 3.10. Let $K > 0$. An orientation-preserving homeomorphism $\phi : U \to V$ between two non-empty, open, connected subsets U, V in $\mathbb{C}$ is said to be *K-quasiconformal* if one has

$$\mathrm{mod}\,\big(\phi(Q)\big) \leq K\,\mathrm{mod}(Q)$$

for every Jordan quadrilateral Q with $\overline{Q} \subseteq U$ (we use the notation $Q \subset\subset U$ for short).

If we cyclically permute the vertices of Q, we automatically obtain

$$\frac{1}{\mathrm{mod}\,\big(\phi(Q)\big)} \leq K\frac{1}{\mathrm{mod}(Q)},$$

which forces $K \geq 1$. This also shows that ϕ^{-1} is K-quasiconformal. By the remark after Definition 3.9, we see that if ϕ is conformal, then it is 1-quasiconformal. We will show the converse of this statement later on. It is also worth noting that the composition of a K-quasiconformal map with a K'-quasiconformal map is a KK'-quasiconformal map. The next proposition gives an alternate definition to Definition 3.9.

Proposition 3.11 (Alternate definition of modulus)**.** *Let Q be a Jordan quadrilateral with vertices p_1, p_2, p_3, p_4. Then $\mathrm{mod}(Q)$ is the smallest quantity with the following property: for any Borel measurable $\rho : Q \to [0, \infty)$ there exists a rectifiable curve γ in Q connecting one a-side of Q to another such that*

$$\left(\int_\gamma \rho(z)|dz|\right)^2 \leq \mathrm{mod}(Q)\int_Q \rho^2(z)dxdy.$$

More compactly,

$$\operatorname{mod}(Q) = \sup_{\rho} \inf_{\gamma} \frac{\left(\int_{\gamma} \rho(z)|dz|\right)^2}{\int_Q \rho^2(z)dxdy}.$$

If we look back at the proof of Theorem 3.3, a similar inequality was a crucial in bounding the length of the interior curve. Now for the proof.

Proof. If $\phi : Q \to Q'$ is a vertex-preserving conformal map between Jordan quadrilaterals Q and Q' with γ a rectifiable curve connecting one a-side of Q to another, we see that $\phi \circ \gamma$ is a rectifiable curve connecting one a-side of Q' to another. By change of variables, we have the following:

$$\int_{\phi\circ\gamma} \rho \circ \phi^{-1}(z)|dz| = \int_{\gamma} \rho(z)|\phi'(z)||dz|,$$
$$\int_{Q'} (\rho \circ \phi^{-1})^2(z)dxdy = \int_Q \rho^2(z)|\phi'(z)|^2 dxdy.$$

So if the proposition holds for Q, it also holds for Q'. Thus without loss of generality, we may assume $Q = (0, M) \times (0, 1)$ where $M = \operatorname{mod}(Q)$ by Theorem 3.8. For any measurable $\rho : Q \to [0, \infty)$, we have by Cauchy-Schwarz and Fubini's theorem that

$$\int_0^1 \left(\int_0^M \rho(x + iy)dx\right)^2 dy \leq M \int_0^1 \int_0^M \rho^2(x + iy)dxdy = M \int_Q \rho^2(z)dxdy.$$

Thus there exists $y \in (0, 1)$ such that

$$\left(\int_0^M \rho(x + iy)dx\right)^2 \leq M \int_Q \rho^2(z)dxdy.$$

On the other hand, if we set $\rho = 1$, then $\int_Q \rho^2(z)dxdy = M$, and for any curve γ connecting an a-side of Q to another (we pick the side on the imaginary axis to be one a-side), we have

$$\int_{\gamma} \rho(z)|dz| = \operatorname{length}(\gamma) \geq M,$$

and hence

$$\left(\int_{\gamma} \rho(z)|dz|\right)^2 \geq M \int_Q \rho^2(z)dxdy.$$

Thus M is the smallest constant with the required property, so we are done. □

We prove some results that follow easily from Proposition 3.11.

Proposition 3.12 (Rengel's Inequality)**.** *Let Q be a Jordan quadrilateral with Euclidean area A. If a is the shortest Euclidean distance from a point on a b-side to another point on the other b-side, and b is the shortest Euclidean distance from a point on a a-side to another point on the other a-side, we have*

$$\frac{b^2}{A} \leq \operatorname{mod}(Q) \leq \frac{A}{a^2}$$

with equality in either case if and only if Q is a rectangle.

Proof. If we set $\rho = 1$ in Proposition 3.11, there exists a rectifiable γ in Q connecting a-sides such that

$$b^2 \leq \left(\int_\gamma |dz| \right)^2 \leq \mathrm{mod}(Q) \int_Q dxdy = \mathrm{mod}(Q)A.$$

To get the other inequality, we apply a cyclic permutation to the vertices to obtain

$$\frac{a^2}{A} \leq \frac{1}{\mathrm{mod}(Q)}.$$

For the equality statement, if Q is a rectangle, equality is clear since $A = ab$ and $\mathrm{mod}(Q) = b/a$. Conversely, suppose $b^2/A = \mathrm{mod}(Q)$ (if we assume the other, we may apply a cyclic permutation to the vertices to obtain this one). Let $\phi : R \to Q$ be a vertex-preserving conformal map where $R = (0, M) \times (0, 1)$ with $M = \mathrm{mod}(Q)$ (via Theorem 3.8). By change of variables, we have

$$A = \int_R |\phi'(z)|^2 dxdy$$

and hence by assumption and Cauchy-Schwarz

$$b^2 = \int_R dxdy \int_R |\phi'(z)|^2 dxdy \geq \left(\int_R |\phi'(z)| dxdy \right)^2. \tag{3.13}$$

However, we also have by Fubini's theorem and the fundamental theorem of calculus that

$$\begin{aligned} \int_R |\phi'(z)| dxdy &= \int_0^1 \int_0^M |\phi'(x+iy)| dxdy \\ &\geq \int_0^1 \left| \int_0^M \frac{\partial \phi}{\partial x}(x+iy) dx \right| dy \\ &= \int_0^1 |\phi(M+iy) - \phi(iy)| dy \geq b \qquad (3.14) \end{aligned}$$

where the last inequality we use the fact that $|\phi(M + iy) - \phi(iy)| \geq b$ since $\phi(iy), \phi(M + iy)$ are on opposite a-sides. Combining (3.13) and (3.14), we have

$$\left(\int_R |\phi'(z)| dxdy \right)^2 = \int_R dxdy \int_R |\phi'(z)|^2 dxdy,$$

so by the equality statement of Cauchy-Schwarz, $|\phi'|$ is constant. Hence ϕ is affine, and thus Q is a rectangle. □

We now prove superadditivity of the modulus.

Proposition 3.15 (Superadditivity)**.** *If Q_1, Q_2 are disjoint Jordan quadrilaterals that share a common a-side, then the Jordan quadrilateral $Q_1 \cup Q_2$ has the property*

$$\mathrm{mod}(Q_1 \cup Q_2) \geq \mathrm{mod}(Q_1) + \mathrm{mod}(Q_2).$$

Moreover, if equality occurs and $Q_1 \cup Q_2$ is mapped by a vertex-preserving conformal map to a rectangle, then Q_1, Q_2 are mapped to sub-rectangles. Similarly, if Q_1, Q_2 share a common b-side, we can perform a cyclic relabeling of the vertices to obtain

$$\frac{1}{\mathrm{mod}(Q_1 \cup Q_2)} \geq \frac{1}{\mathrm{mod}(Q_1)} + \frac{1}{\mathrm{mod}(Q_2)}.$$

Proof. If we map $Q := Q_1 \cup Q_2$ to the rectangle $R = (0, M) \times (0, 1)$ with a vertex-preserving conformal map, we see that there exists a curve connecting b-sides of R, partitioning it into two Jordan quadrilaterals R_1 and R_2. By definition, we have

$$\text{area}(R_1) + \text{area}(R_2) = \text{area}(R) = \text{mod}(Q). \tag{3.16}$$

Since our vertex-preserving map is a homeomorphism, it sends connected components to connected components, so without loss of generality we may assume Q_1, Q_2 get mapped to R_1, R_2 respectively (in a vertex-preserving fashion). By Proposition 3.12, we have

$$\text{area}(R_1) \geq \text{mod}(Q_1) \quad \text{and} \quad \text{area}(R_2) \geq \text{mod}(Q_2). \tag{3.17}$$

Thus by combining (3.16) and (3.17), we have the desired inequality. The equality statement follows from the equality statement of Proposition 3.12. $\square$

Now we prove the converse of an earlier statement.

Proposition 3.18. *Every* 1*-quasiconformal map* $\phi : U \to V$ *is conformal.*

Proof. Suppose $Q \subset\subset U$ is a Jordan quadrilateral. By composing ϕ with vertex-preserving conformal maps on the left and right, we may assume both Q and $\phi(Q)$ are rectangles. Since ϕ is 1-conformal, we can further assume $Q = \phi(Q) = (0, M) \times (0, 1)$ where $M = \text{mod}(Q)$. If we subdivide Q into two rectangles by a vertical line segment $\{x\} \times (0, 1)$ for a fixed $x \in (0, M)$, the moduli of these rectangles are x and $M - x$ respectively. Applying ϕ and using Proposition 3.15, we have that ϕ preserves these rectangles and hence preserves the x-coordinate. Similarly, we do this process with a horizontal line segment, and we see ϕ preserves the y-coordinate, and is thus the identity, which is clearly conformal. $\square$

We next give an equivalent definition K-quasiconformal in terms of directional derivatives. We first recall some facts from linear algebra. Suppose $\phi : U \to V$ is an orientation preserving diffeomorphism between open, connected sets $U, V \subseteq \mathbb{C}$. Then we have for $z_0 \in U$ and $h \in \mathbb{C}$ sufficiently small

$$\phi(z_0 + h) = \phi(z_0) + \frac{\partial \phi}{\partial z}(z_0)h + \frac{\partial \phi}{\partial \overline{z}}(z_0)\overline{h} + o(h) \tag{3.19}$$

where

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and $o(h)$ is a quantity such that $o(h)/|h| \to 0$ as $h \to 0$. We can do this because any linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ can be represented as $az + b\overline{z}$ for some $a, b \in \mathbb{C}$ where we use the ring isomorphism

$$a + bi \leftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

identifying $\mathbb{C}$ with a subspace of $\mathcal{M}_{2\times 2}(\mathbb{R})$. Furthermore, if $A = az + b\overline{z}$, then through computation we see

$$\det(A) = |a|^2 - |b|^2.$$

Thus, since ϕ is orientation-preserving, we have

$$\det D\phi(z_0) = \left|\frac{\partial \phi}{\partial z}(z_0)\right|^2 - \left|\frac{\partial \phi}{\partial \overline{z}}(z_0)\right|^2 > 0.$$

Recall from singular value decomposition, there exists orthonormal bases (e_1, e_2) and (f_1, f_2) in $\mathbb{C}$ with positive scalars s_1, s_2 such that

$$D\phi(z_0)e_1 = s_1 f_1 \quad \text{and} \quad D\phi(z_0)e_2 = s_2 f_2$$

with

$$\det D\phi(z_0) = |\det D\phi(z_0)| = s_1 s_2 > 0. \tag{3.20}$$

Suppose, without loss of generality, that $s_1 \geq s_2$. If $D_u\phi(z) := \frac{d}{dt}\phi(z+ut)\big|_{t=0}$ denotes the directional derivative, it can be seen that

$$\sup_{|u|=1} |D_u\phi(z_0)| = \sup_{|u|=1} |D\phi(z_0)u| = s_1$$
$$\inf_{|u|=1} |D_u\phi(z_0)| = \inf_{|u|=1} |D\phi(z_0)u| = s_2$$

by viewing the linear transformation as mapping the unit circle to an ellipse with radii s_1, s_2. On the other hand, using (3.19), we see that by choosing suitable values for u and the triangle inequality that

$$\begin{aligned} \sup_{|u|=1} |D_u\phi(z_0)| &= \sup_{|u|=1} \left| \frac{\partial \phi}{\partial z}(z_0)u + \frac{\partial \phi}{\partial \overline{z}}(z_0)\overline{u} \right| = \left| \frac{\partial \phi}{\partial z}(z_0) \right| + \left| \frac{\partial \phi}{\partial \overline{z}}(z_0) \right| \\ \inf_{|u|=1} |D_u\phi(z_0)| &= \inf_{|u|=1} \left| \frac{\partial \phi}{\partial z}(z_0)u + \frac{\partial \phi}{\partial \overline{z}}(z_0)\overline{u} \right| = \left| \frac{\partial \phi}{\partial z}(z_0) \right| - \left| \frac{\partial \phi}{\partial \overline{z}}(z_0) \right|. \end{aligned} \tag{3.21}$$

Note that both of these are positive in this case since $s_1, s_2 > 0$. We will use these relations extensively in the next two statements.

Theorem 3.22. *Let $K \geq 1$ and $\phi : U \to V$ be an orientation preserving diffeomorphism between open, connected sets $U, V \subseteq \mathbb{C}$. Then ϕ being K-quasiconformal is equivalent to the property that for any $z \in U$ and $u, v \in S^1 := \{z \in \mathbb{C} \;;\; |z| = 1\}$ we have*

$$|D_u\phi(z)| \leq K|D_v\phi(z)|.$$

Proof. We first prove that the desired property holds if ϕ is K-quasiconformal by contrapositive. So there exists $z_0 \in U$ such that

$$\sup_{|u|=1} |D_u\phi(z_0)| > K \inf_{|u|=1} |D_u\phi(z_0)| > 0.$$

So by singular value decomposition, there exists $v \in S^1$ such that

$$D_{Rv}\phi(z_0) = \lambda R\big[D_v\phi(z_0)\big] \tag{3.23}$$

where $\lambda > K$ and $R : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation matrix of $\pi/2$. For z sufficiently small, we define

$$\psi(z) := \frac{\phi(z_0 + zv) - \phi(z_0)}{D_v\phi(z_0)}.$$

So we have by definition and (3.23) that

$$\psi(0) = 0,\ \frac{\partial \psi}{\partial x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \frac{\partial \psi}{\partial y}(0) = \begin{pmatrix} 0 \\ \lambda \end{pmatrix}$$

By the linearization of ψ, we have for sufficiently small h that

$$\psi(h) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} h + o(h) \tag{3.24}$$

where $o(h)$ is a quantity such that $o(h)/|h| \to 0$ as $h \to 0$. If we let $\varepsilon > 0$ be sufficiently small and $R = (0, \varepsilon) \times (0, \varepsilon)$, we see that by (3.24) and Proposition 3.12, we have that $\operatorname{mod}\big(\psi(R)\big) > K \operatorname{mod}(R) = K$, which implies ψ is not K-quasiconformal and hence ϕ is not K-quasiconformal.

Conversely, suppose $Q \subset\subset U$ is a Jordan quadrilateral and ϕ satisfies the property that for any $z \in U$ and $u, v \in S^1 := \{z \in \mathbb{C} \ ; \ |z| = 1\}$ we have

$$|D_u \phi(z)| \leq K |D_v \phi(z)|.$$

One can check that this property is unchanged by composing ϕ with conformal maps on the left or right, so we may assume that $Q = (0, M) \times (0, 1)$ and $\phi(Q) = (0, M') \times (0, 1)$ where $M := \operatorname{mod}(Q)$ and $M' := \operatorname{mod}\big(\phi(Q)\big)$. Then by change of variables and Fubini's theorem, we have

$$\begin{aligned}
M' &= \int_{\phi(Q)} dxdy \\
&= \int_Q \det D\phi(z) dxdy \\
&\overset{(3.20)}{=} \int_Q \sup_{|u|=1} |D_u \phi(z)| \inf_{|u|=1} |D_u \phi(z)| dxdy \\
&\geq \int_Q \frac{1}{K} \left| \frac{\partial \phi}{\partial x}(z) \right|^2 dxdy \\
&= \frac{1}{K} \int_0^1 \int_0^M \left| \frac{\partial \phi}{\partial x}(x + iy) \right|^2 dxdy \\
&\geq \frac{1}{MK} \int_0^1 \left| \int_0^M \frac{\partial \phi}{\partial x}(x + iy) dx \right|^2 dy \\
&\geq \frac{1}{MK} \int_0^1 (M')^2 dy
\end{aligned}$$

where the penultimate inequality we use the Cauchy-Schwarz inequality. This gives $M' \leq KM$, and thus ϕ is K-quasiconformal. $\square$

Corollary 3.25. *Let $K \geq 1$ and $\phi : U \to V$ be an orientation preserving diffeomorphism between open, connected sets $U, V \subseteq \mathbb{C}$. Then ϕ being K-quasiconformal is equivalent to*

$$\left| \frac{\partial \phi}{\partial \overline{z}}(z) \right| \leq \frac{K-1}{K+1} \left| \frac{\partial \phi}{\partial z}(z) \right|$$

for all $z \in U$.

Proof. The property described in Theorem 3.22 is equivalent to

$$\sup_{|u|=1} |D_u \phi(z)| \leq K \inf_{|v|=1} |D_v \phi(z)|,$$

for all $z \in U$. By (3.21), this is equivalent to the desired result. $\square$

The next few statements cover extending quasiconformal maps to larger domains. This includes analogs of Riemann's removable singularity theorem and the Schwartz reflection principle. To this end, we next prove a very technical property of quasiconformal maps.

Proposition 3.26 (Absolute Continuity on Lines)**.** *Let $\phi : U \to V$ be a K-quasiconformal map between two open, connected subsets of $\mathbb{C}$. If $(0, M) \times (0, 1) \subset\subset U$, then for almost every $t \in [0, 1]$, the function $\phi(\cdot + ti)$ is absolutely continuous on $[0, M]$.*

Proof. We first define

$$A(t) = \text{area}\Big(\phi\big([0, M] \times [0, t]\big)\Big)$$

for $t \in [0, 1]$. We see that A is a non-decreasing function and hence differentiable almost everywhere. We claim that for $t \in [0, 1]$ where A is differentiable, $\phi(\cdot + ti)$ is absolutely continuous on $[0, M]$. Indeed, let $\varepsilon > 0$. Then we let $\delta := \frac{\varepsilon^2}{KA'(t)}$ if $A'(t) \neq 0$ (if $A'(t) = 0$, we can make δ anything greater than 0). Suppose $[x_1, y_1], \ldots, [x_m, y_m]$ are disjoint intervals in $[0, M]$ with $\sum_{j=1}^m (y_j - x_j) \leq \delta$. Now let $\eta > 0$ be a small number that can depend on these intervals. We define

$$R_j := (x_j, y_j) \times (t, t + \eta)$$

for $j = 1, \ldots, m$. Note $\text{mod}(R_j) = \frac{y_j - x_j}{\eta}$, and since ϕ is K-quasiconformal, we have that

$$\text{mod}\big(\phi(R_j)\big) \leq K\frac{y_j - x_j}{\eta}. \tag{3.27}$$

On the other hand, using Proposition 3.12, we have

$$\frac{|\phi(y_j + ti) - \phi(x_j + ti) - o(1)|^2}{\text{area}\big(\phi(R_j)\big)} \leq \text{mod}\big(\phi(R_j)\big) \tag{3.28}$$

where $o(1)$ is a quantity with the property that $o(1) \to 0$ as $\eta \to 0$. Combining (3.27) with (3.28) and summing over $j = 1, \ldots, m$, we see

$$\sum_{j=1}^m |\phi(y_j + ti) - \phi(x_j + ti) - o(1)| \leq \sqrt{\frac{K}{\eta}}\left(\sum_{j=1}^m \sqrt{y_j - x_j}\sqrt{\text{area}\big(\phi(R_j)\big)}\right)$$

Squaring and using the Cauchy-Schwarz inequality, we have

$$\left(\sum_{j=1}^m |\phi(y_j + ti) - \phi(x_j + ti) - o(1)|\right)^2 \leq KA'(t)\sum_{j=1}^m (y_j - x_j) + o(1).$$

where we use the fact that $\sum_{j=1}^m \text{area}\big(\phi(R_j)\big) \leq A(t + \eta) - A(t) = (A'(t) + o(1))\eta$. Finally, sending $\eta \to 0$, we have

$$\sum_{j=1}^m |\phi(y_j + ti) - \phi(x_j + ti)| \leq \sqrt{KA'(t)\delta} < \varepsilon,$$

which concludes the proof. $\square$

The next statement uses this proposition in a crucial part of the proof.

Proposition 3.29. *Let $\phi : U \to V$ be a K-quasiconformal map between two open, connected subsets of $\mathbb{C}$. If $N \subset \mathbb{C}$ is a closed set of Lebesgue measure 0 such that ϕ is conformal on $U \setminus N$, then ϕ is 1-quasiconformal, and hence by Proposition 3.18, ϕ is conformal.*

Proof. Suppose $Q \subset\subset U$ is a Jordan quadrilateral. Note that the image of a Lebesgue measure zero sets under conformal maps is Lebesgue measure zero, since conformal maps are locally Lipschitz (in literature, this condition is called *Lusin's N-condition*). So, by composing ϕ with conformal maps on the left and right, we may assume $Q = (0, M) \times (0, 1)$ and $\phi(Q) = (0, M') \times (0, 1)$ where $M = \mathrm{mod}(Q)$ and $M' = \mathrm{mod}\big(\phi(Q)\big)$ (with N still having Lebesgue measure 0). Following the proof of Theorem 3.22, we have that by change of variables and Fubini's theorem

$$\begin{aligned}
M' &\geq \int_{\phi(Q\setminus N)} dxdy \\
&= \int_{Q\setminus N} \left|\frac{\partial \phi}{\partial x}(z)\right|^2 dxdy \\
&= \int_{Q} \left|\frac{\partial \phi}{\partial x}(z)\right|^2 dxdy \\
&= \int_0^1 \int_0^M \left|\frac{\partial \phi}{\partial x}(x+iy)\right|^2 dxdy \\
&\geq \frac{1}{M}\int_0^1 \left|\int_0^M \frac{\partial \phi}{\partial x}(x+iy)dx\right|^2 dy \\
&= \frac{1}{M}\int_0^1 |\phi(M+yi) - \phi(yi)|^2 dy \\
&\geq \frac{1}{M}\int_0^1 (M')^2 dy
\end{aligned}$$

where the penultimate inequality we use Cauchy-Schwarz and the last equality we use the Lebesgue's fundamental theorem of calculus (utilizing Proposition 3.26). This gives $M' \leq M$, and thus ϕ is 1-quasiconformal. $\square$

Recall Hurwitz's theorem which states that the locally uniform limit of conformal maps is either conformal or constant. The next theorem is a weaker form of this for quasiconformal maps. Before we do this, we state and prove a continuity property of the modulus.

Proposition 3.30. *Suppose Q_n is a sequence of Jordan quadrilaterals which converge to another Jordan quadrilateral Q where $Q_n \subseteq Q$. By this, we mean the vertices of Q_n converge to their respective counterparts in Q and each a-side in Q_n converges to the a-side of Q in the Hausdorff sense, and similarly for the b-sides. Then we have*

$$\mathrm{mod}(Q_n) \to \mathrm{mod}(Q)$$

as $n \to \infty$.

Proof. Since the Jordan quadrilateral convergence is preserved under vertex-preserving conformal maps, we may assume $Q = (0, M) \times (0, 1)$ where $M = \mathrm{mod}(Q)$. By Proposition 3.12, we have for all $n \geq 1$ that

$$\frac{b_n^2}{A_n} \leq \mathrm{mod}(Q_n) \leq \frac{A_n}{a_n^2} \tag{3.31}$$

where A_n is the area of Q_n, a_n is the shortest Euclidean distance from a point on a b-side to another point on the other b-side of Q_n, and b_n is the shortest Euclidean distance from a point on a a-side to another point on the other a-side of Q_n. By the definition of Hausdorff convergence, for any $\varepsilon > 0$ there exists $N > 0$ such that for all $n \geq N$ we have the Hausdorff distances between the corresponding sides is less than ε with

$$(\varepsilon, M - \varepsilon) \times (\varepsilon, 1 - \varepsilon) \subseteq Q_n.$$

Thus we have

$$\begin{aligned} A_n &\leq M \\ M - 2\varepsilon &\leq b_n \\ 1 - 2\varepsilon &\leq a_n, \end{aligned}$$

which, with (3.31), implies

$$\frac{(M-2\varepsilon)^2}{M} \leq \operatorname{mod}(Q_n) \leq \frac{M}{(1-2\varepsilon)^2}$$

for $\varepsilon < 1/2$. Thus $\operatorname{mod}(Q_n) \to \operatorname{mod}(Q)$ as $n \to \infty$. □

Now for Hurwitz's theorem for quasiconformal maps.

Theorem 3.32 (Hurwitz)**.** *Let $K \geq 1$ and $\phi_n : U \to \phi_n(U)$ be a sequence of K-quasiconformal maps that converge locally uniformly to an orientation-preserving homeomorphism $\phi : U \to V$. Then ϕ is also K-quasiconformal.*

Proof. Let $Q \subset\subset U$ be a Jordan quadrilateral. By composing the ϕ_n's and ϕ with a conformal map, we may assume $Q = (0, M) \times (0, 1)$ where $M = \operatorname{mod}(Q)$. Now let

$$Q_m := \left(\frac{1}{m}, M - \frac{1}{m}\right) \times \left(\frac{1}{m}, 1 + \frac{M-2}{mM}\right)$$

for $m > 2/M$. It is easy to see that $\operatorname{mod}(Q_m) = \operatorname{mod}(Q)$, and so

$$\operatorname{mod}\big(\phi_n(Q_m)\big) \leq K \operatorname{mod}(Q) \tag{3.33}$$

for all $m > 2/M$ and $n \geq 1$. Now for all $m > 2/M$, $\overline{Q_m}$ is compact and disjoint from $\mathbb{C} \setminus \phi(Q)$, which is closed, so there exists $\varepsilon_m > 0$ such that

$$|\phi(z) - w| \geq \varepsilon_m \tag{3.34}$$

for all $z \in \overline{Q_m}$ and $w \in \mathbb{C} \setminus \phi(Q)$. On the other hand, we have $\phi_n \to \phi$ uniformly on $\overline{Q_m}$ and so there exists $n_m > 0$ where

$$|\phi_{n_m}(z) - \phi(z)| < \frac{\varepsilon_m}{2} \tag{3.35}$$

for all $z \in \overline{Q_m}$. Combining (3.34) and (3.35), by the triangle inequality we have

$$|\phi_{n_m}(z) - w| \geq \frac{\varepsilon_m}{2}$$

for all $z \in \overline{Q_m}$ and $w \in \mathbb{C} \setminus \phi(Q)$. Hence $\phi_{n_m}(Q_m) \subseteq \phi(Q)$. Since $Q_m \nearrow Q$ in the sense of Proposition 3.30 and $\phi_{n_m} \to \phi$ locally uniformly, we have $\phi_{n_m}(Q_m) \to \phi(Q)$ in the sense of Proposition 3.30. Thus $\operatorname{mod}\big(\phi_{n_m}(Q_m)\big) \to \operatorname{mod}\big(\phi(Q)\big)$. Substituting n_m for n in (3.33), we have our result after letting $m \to \infty$. □

We next prove the quasiconformal analog to Riemann's removable singularity theorem.

Theorem 3.36. *Suppose $\phi : U \to V$ is K-quasiconformal and z_0 is an isolated boundary point of U. Suppose there is an orientation-preserving homeomorphism $\widetilde{\phi}$ that is an extension of ϕ on $U \cup \{z_0\}$. Then $\widetilde{\phi}$ is K-quasiconformal.*

Note that the assumption of the existence of an extension can be dropped (see Theorem 8.1 on page 47 of [10]). We will, however, be only using this statement.

Proof. Let $Q \subset\subset U \cup \{z_0\}$ be a Jordan quadrilateral. We can assume $z_0 \in Q$ for otherwise the statement would follow from the K-quasiconformality of ϕ. By composing $\widetilde{\phi}$ with a conformal map, we may assume $\widetilde{\phi}(Q) = (0, M') \times (0,1)$. Now consider $Q_1' := (0, \Re\widetilde{\phi}(z_0)) \times (0,1)$ and $Q_2' := (\Re\widetilde{\phi}(z_0), M') \times (0,1)$. By Proposition 3.15, we have

$$\begin{aligned}\operatorname{mod}\big(\widetilde{\phi}(Q)\big) &= \operatorname{mod}(Q_1') + \operatorname{mod}(Q_2')\\ &\le K \operatorname{mod}\big(\phi^{-1}(Q_1')\big) + K \operatorname{mod}\big(\phi^{-1}(Q_2')\big)\\ &\le K \operatorname{mod}\big(\phi^{-1}(Q_1') \cup \phi^{-1}(Q_2')\big) = K \operatorname{mod}(Q),\end{aligned}$$

as desired. □

Before we prove the analog of Schwarz's reflection principle, we first prove an easy but important consequence of Proposition 3.30. We first start with a definition.

Definition 3.37. An injective curve $\gamma : [0,1] \to \mathbb{C}$ is called *analytic* if it is the image of $[0,1]$ under a conformal map defined in a neighborhood of $[0,1]$. Similarly, a Jordan curve $\gamma : S^1 \to \mathbb{C}$ is called *analytic* if it is the image of S^1 under a conformal map defined in a neighborhood of S^1.

Lemma 3.38. *Let $\phi : U \to V$ be an orientation-preserving homeomorphism that satisfies*

$$\operatorname{mod}\big(\phi(Q)\big) \le K \operatorname{mod} Q$$

for all Jordan quadrilaterals $Q \subset\subset U$ with analytic sides whose mapping to a rectangle (from Theorem 3.8) has a conformal extension to a domain containing $\overline{Q}$. Then ϕ is K-quasiconformal.

Proof. Let $Q \subset\subset U$ be a Jordan quadrilateral. By Theorem 3.8, there exists a conformal map f taking Q to a rectangle $R = (0, M) \times (0,1)$. We approximate R from the inside by

$$R_n := \left(\frac{1}{n}, M - \frac{1}{n}\right) \times \left(\frac{1}{n}, 1 - \frac{1}{n}\right).$$

Consider $Q_n := f^{-1}(R_n)$. These Jordan quadrilaterals approximate Q in the sense of Proposition 3.30 by the uniform continuity of f's extension to $\overline{Q}$. Furthermore, each canonical map of Q_n has a conformal extension (notably f), and each side of Q_n is analytic. By assumption, we have

$$\operatorname{mod}\big(\phi(Q_n)\big) \le K \operatorname{mod} Q_n$$

for all n. Letting $n \to \infty$ and using the uniform continuity of ϕ, we have the desired result. □

Now we are set to prove the analog to Schwartz's reflection principle for quasiconformal maps.

Theorem 3.39 (Schwartz)**.** *Let $K \ge 1$ and $\phi : U \to V$ be an orientation-preserving homeomorphism. Suppose $\gamma : [0,1] \to \overline{U}$ is an analytic curve lying in U except possibly at its endpoints. Then if $\phi : U \setminus \gamma \to \phi(U \setminus \gamma)$ is K-quasiconformal, then $\phi : U \to V$ is K-quasiconformal.*

Proof. Let $Q \subset\subset U$ be a Jordan quadrilateral with analytic sides whose mapping to a rectangle has a conformal extension. Since each side of Q is analytic, we have that $\overline{Q} \cap \gamma$ is a finite union of disjoint closed analytic arcs $\gamma_1, \ldots, \gamma_n$. By composing ϕ with conformal maps, we may assume $Q = (0, M) \times (0, 1)$ and $\phi(Q) = (0, M') \times (0, 1)$ where $M = \mathrm{mod}(Q)$ and $M' = \mathrm{mod}(Q')$. If γ_j is not a horizontal segment, since it is analytic, it can be divided into finitely many curves which intersect any horizontal line at most once (this can be done by dividing at the finitely many points where $\Im \gamma_j' = 0$). We can partition Q into horizontal rectangles

$$R_k := (0, M) \times (y_{k-1}, y_k)$$

where $0 = y_0 < y_1 < \cdots < y_m = 1$ and such that for every k, j, $R_k \cap \gamma_j$ is a finite union of curves joining b-sides of R_k. Note that if we refine this partition, the maximum number of curves intersecting an R_k is invariant. So for any $\varepsilon > 0$, using the uniform continuity of ϕ, we can find a refinement of the partition above such that for every k, the sum of the diameters of the components of $\phi\big(R_k \cap (\bigcup_j \gamma_j)\big)$ is less than ε. Without loss of generality, we call this refinement the same as the partition above. Note that in this partition, the γ_j divide each R_k into finitely many (possibly degenerate) quadrilaterals $R_{k,h}$. Denote the images of R_k and $R_{k,h}$ in Q' by R_k' and $R_{k,h}'$ respectively. If $d_{k,h}$ denotes the distance between the a-sides of $R_{k,h}'$, then by the property gained from the refinement we have

$$M' - \sum_h d_{k,h} < \varepsilon \tag{3.40}$$

For the nondegenerate Jordan quadrilateral $R_{k,h}'$, we have by Proposition 3.12 that

$$\frac{d_{k,h}^2}{\mathrm{area}(R_{k,h}')} \le \mathrm{mod}(R_{k,h}').$$

By summing over h and using $\mathrm{mod}(R_{k,h}') \le K \,\mathrm{mod}(R_{k,h})$, we have

$$\begin{aligned}
\frac{1}{K} \sum_h \frac{d_{k,h}^2}{\mathrm{area}(R_{k,h}')} &= \frac{1}{K} \sum_{h: R_{k,h}' \text{ nondeg.}} \frac{d_{k,h}^2}{\mathrm{area}(R_{k,h}')} \\
&\le \frac{1}{K} \sum_{h: R_{k,h}' \text{ nondeg.}} \mathrm{mod}(R_{k,h}') \\
&\le \sum_{h: R_{k,h}' \text{ nondeg.}} \mathrm{mod}(R_{k,h}) \\
&\overset{3.12}{\le} \sum_{h: R_{k,h}' \text{ nondeg.}} \mathrm{area}(R_{k,h}) \\
&\le \sum_h \mathrm{area}(R_{k,h}) \\
&= \mathrm{area}(R_k) = \mathrm{mod}(R_k).
\end{aligned}$$

and hence by Cauchy-Schwarz

$$\mathrm{mod}(R_k) \ge \frac{1}{K} \frac{\big(\sum_h d_{k,h}\big)^2}{\sum_h \mathrm{area}(R_{k,h}')}.$$

By (3.40), we have

$$\operatorname{mod}(R_k) \geq \frac{1}{K}\frac{(M'-\varepsilon)^2}{\operatorname{area}(R_k')} \implies \operatorname{area}(R_k') \geq \frac{1}{K}\frac{(M'-\varepsilon)^2}{\operatorname{mod}(R_k)}$$

and summing over k and using Proposition 3.15, we obtain

$$M' \geq \frac{1}{K}\frac{(M'-\varepsilon)^2}{M}$$

After we let $\varepsilon \to 0$, we are done. ☐

We next define ring domains and their complex modulus.

Definition 3.41. Given any two Jordan curves γ_1, γ_2 where γ_1 is contained in the interior of γ_2, we define the *ring domain* between γ_1 and γ_2 to be the open region between these curves. An example of a ring domain is an annulus $A_{r,R} := \{z \in \mathbb{C} \; ; \; r < |z| < R\}$.

We give regularity to this definition with the following theorem.

Theorem 3.42. *For every ring domain A there exists a conformal map $\psi : A \to A_{1,R}$ for some $R > 1$. Furthermore, this R is unique.*

Proof of Existence. Let A be a ring domain with boundary Jordan curves γ_1, γ_2 with γ_1 in the interior of γ_2. Since the interior of a Jordan curve is simply connected, by applying a Riemann map, we can assume γ_2 is the unit circle and γ_1 is a Jordan curve in D with 0 in it's interior. From this, we can see A is homeomorphic to the exterior of γ_1, which implies A has fundamental group $\mathbb{Z}$. Recall that the exponential map $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a covering map. So $A' := \exp^{-1} A$ is connected. Furthermore, A' is simply connected. One can see this because by covering space theory we have

$$\exp^{-1}\{a\} \cong \frac{\pi_1(A, a)}{\exp_* \pi_1(A', a')}$$

where $e^{a'} = a$ for some $a \in A$. Now $\exp^{-1}\{a\}$ is infinite while $\pi_1(A, a) = \mathbb{Z}$, so this implies $\exp_* \pi_1(A', a') = 0$ and hence $\pi_1(A', a') = 0$. It is easy to see that A' is not the whole plane since $0 \notin A'$. By the Riemann mapping theorem, there exists a biholomorphic map $f : A' \to D$. Fix $a' \in A'$, and consider the sequences

$$\begin{aligned} a_n^+ &:= a' + 2\pi i n \\ a_n^- &:= a' - 2\pi i n. \end{aligned}$$

It is clear $a_n^+, a_n^- \in A'$, and since f is proper, $f(a_n^+)$ and $f(a_n^-)$ escape to infinity in D. Since the sequences $f(a_n^+)$ and $f(a_n^-)$ are bounded, there exists subsequences of both converging to $\theta^+, \theta^- \in \partial D$ respectively (note $\theta^+ \neq \theta^-$ by continuity). We claim that the original sequences $f(a_n^+)$ and $f(a_n^-)$ converge to θ^+ and θ^- respectively. Indeed, this follows from the continuity of $f(\frac{1}{z})$ on $\frac{1}{A}$. By composing f with an automorphism of the unit disc, we can assume $\theta^+ = 1$ and $\theta^- = -1$. Now consider $S := \{z \mid -\pi < \Re(z) < 0\}$ and $g : D \to S$ where

$$g(z) := i \log \frac{z+1}{i(z-1)}$$

with log being the principal branch. If $h := g \circ f$, then it is easy to see that $\Im(h(a_n^+)) \to \infty$ and $\Im(h(a_n^-)) \to -\infty$. Note that the function $h' : A' \to S$ where

$h'(z) := h(z + 2\pi i)$ has this same property. So $h' \circ h^{-1}$ is an automorphism of S that takes a sequence diverging to $i\infty$ to a sequence diverging to $i\infty$ and similarly for $-i\infty$. Note that the automorphism group of S is

$$\Big\{ i \log \frac{ae^{-iz} + b}{ce^{-iz} + d} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1 \Big\}.$$

When looking at $h' \circ h^{-1}$, the statement about $i\infty$ forces $c = 0$, and the statement about $-i\infty$ forces $b = 0$. Thus $h' \circ h^{-1}(z) = z + i \log a^2$ where $a \in \mathbb{R}_{\neq 0}$. In other words,

$$h(z + 2\pi i) = h(z) + yi \implies \frac{2\pi}{y} h(z + 2\pi i) = \frac{2\pi}{y} h(z) + 2\pi i$$

for some $y \in \mathbb{R}_{\neq 0}$. So we have found a biholomorphic map $\frac{2\pi}{y} h$ from A' to a vertical strip that is invariant under adding $2\pi i$. Then it is easy to see that the map

$$e^{\frac{2\pi}{y} h(\log z)}$$

is a well-defined biholomorphic map from A to an annulus. Scaling if necessary, we can assume the annulus is $A_{1,R}$ for some $R > 1$. □

Proof of Uniqueness. Suppose we have too such maps $\psi : A \to A_{1,R}$ and $\psi' : A \to A_{1,R'}$. Note $\psi' \circ \psi^{-1} : A_{1,R} \to A_{1,R'}$. By a proof similar to Theorem 3.3, we see $\psi' \circ \psi^{-1}$ extends to a homeomorphism from $\overline{A_{1,R}}$ to $\overline{A_{1,R'}}$. By composing with an inversion, we can assume ∂D maps to to ∂D and ∂D_R maps to $\partial D_{R'}$. By Schwartz reflection principle, we can extend $\psi' \circ \psi^{-1}$ to an automorphism of $\mathbb{C} \setminus \{0\}$. By Riemann's removable singularity theorem, we can finally extend this map to an automorphism of the complex plane. Since $0 \mapsto 0$ and ∂D maps to ∂D, we see that

$$\psi' \circ \psi^{-1}(z) = e^{i\theta} z$$

for some $\theta \in \mathbb{R}$ (or $\psi' \circ \psi^{-1}(z) = \frac{1}{e^{i\theta} z}$ if we had to compose with an inversion). Since ∂D_R maps to $\partial D_{R'}$, we see $R = R'$, as desired. □

Now for the definition of the modulus of a ring domain.

Definition 3.43. Let A be a ring domain. Then we define

$$\mathrm{mod}(A) := \log R$$

where R is as in Theorem 3.42.

Note that this definition agrees (up to a scaling) with the definition of the modulus of a quadrilateral in the following way: if we apply the principal branch of the logarithm to $A_{1,R} \setminus (-\infty, 0]$, we get a rectangle $(0, \log R) \times (-\pi, \pi)$, which has modulus $\frac{1}{2\pi} \log R$. We now prove an equivalence.

Proposition 3.44 (Alternate definition of modulus for ring domains)**.** *Let A be a ring domain. Then we have the following two statements:*

- $\mathrm{mod}(A)$ *is the largest quantity with the following property: for any Borel measurable $\rho : A \to [0, \infty)$ there exists a rectifiable curve γ in A winding once around the inner boundary such that*

$$\left(\int_\gamma \rho(z) |dz| \right)^2 \leq \frac{2\pi}{\mathrm{mod}(A)} \int_A \rho^2(z) dx dy.$$

- $\mathrm{mod}(A)$ *is the smallest quantity with the following property: for any Borel measurable* $\rho : A \to [0, \infty)$ *there exists a rectifiable curve* γ *in* A *connecting the disjoint boundary curves* γ_1, γ_2 *of* A *such that*

$$\left(\int_\gamma \rho(z)|dz| \right)^2 \leq \frac{\mathrm{mod}(A)}{2\pi} \int_A \rho^2(z) dxdy.$$

Proof. Like in the proof of Proposition 3.11, after using Theorem 3.42 and a change of variables, we may assume $A = A_{1,R}$. We first prove the first statement. For this, we first note that given $\rho : A_{1,R} \to [0, \infty)$, there exists $t \in (1, R)$ such that

$$\int_0^{2\pi} t^2 \rho^2(te^{i\theta}) d\theta \leq \frac{1}{\log R} \int_{A_{1,R}} \rho^2(z) dxdy \tag{3.45}$$

for otherwise, we would get a contradiction by dividing by t and integrating both sides of the inequality. Letting $\gamma(\theta) = te^{i\theta}$ for $\theta \in [0, 2\pi)$, we see

$$\begin{aligned}
\left(\int_\gamma \rho(z)|dz| \right)^2 &= \left(\int_0^{2\pi} t\rho(te^{i\theta}) d\theta \right)^2 \\
&\leq 2\pi \int_0^{2\pi} t^2 \rho^2(te^{i\theta}) d\theta \\
&\overset{(3.45)}{\leq} \frac{2\pi}{\log R} \int_{A_{1,R}} \rho^2(z) dxdy
\end{aligned}$$

where the first inequality follows from the Cauchy-Schwarz inequality. On the other hand, if we set $\rho(z) = 1/|z|$, then for any rectifiable curve γ in $A_{1,R}$ winding once around the inner boundary we have

$$\left(\int_\gamma \frac{1}{|z|} |dz| \right)^2 \geq \left| \int_\gamma \frac{1}{z} dz \right|^2 = (2\pi)^2$$

and

$$\int_{A_{1,R}} \frac{1}{|z|^2} dxdy = \int_0^{2\pi} \int_1^R \frac{1}{t} dt d\theta = 2\pi \log R.$$

So we have

$$\left(\int_\gamma \rho(z)|dz| \right)^2 \geq \frac{2\pi}{\log R} \int_{A_{1,R}} \rho^2(z) dxdy.$$

The second statement in the theorem is very similar. Given $\rho : A_{1,R} \to [0, \infty)$, there exists $\theta \in [0, 2\pi)$ such that

$$\int_1^R t\rho^2(te^{i\theta}) dt \leq \frac{1}{2\pi} \int_{A_{1,R}} \rho^2(z) dxdy. \tag{3.46}$$

Letting $\gamma(t) = te^{i\theta}$ for $t \in [1, R]$, we have

$$\begin{aligned}
\left(\int_\gamma \rho(z)|dz| \right)^2 &= \left(\int_1^R \rho(te^{i\theta}) dt \right)^2 \\
&\leq \log R \int_1^R t\rho^2(te^{i\theta}) dt \\
&\overset{(3.46)}{\leq} \frac{\log R}{2\pi} \int_{A_{1,R}} \rho^2(z) dxdy
\end{aligned}$$

where the first inequality follows from the Cauchy-Schwarz inequality. On the other hand, if we set $\rho(z) = 1/|z|$, then for any rectifiable curve γ in $A_{1,R}$ connecting the boundary curves, we have

$$\left(\int_\gamma \frac{1}{|z|}|dz|\right)^2 \geq \left(\int_\gamma \frac{1}{|z|}d|z|\right)^2 = (\log R)^2$$

and

$$\int_{A_{1,R}} \frac{1}{|z|^2}dxdy = \int_0^{2\pi}\int_1^R \frac{1}{t}dtd\theta = 2\pi \log R.$$

So we have

$$\left(\int_\gamma \rho(z)|dz|\right)^2 \geq \frac{\log R}{2\pi}\int_{A_{1,R}} \rho^2(z)dxdy.$$

□

Remark 3.47. It is clear from the first statement of the theorem that if A and B are ring domains with $A \subseteq B$ then we have $\mathrm{mod}(A) \leq \mathrm{mod}(B)$. Indeed, take any arbitrary $\rho : B \to [0,\infty)$. If we consider $\rho|_A$, then there exists a rectifiable $\gamma \subset A \subseteq B$ where

$$\begin{aligned}\left(\int_\gamma \rho(z)|dz|\right)^2 &\leq \frac{2\pi}{\mathrm{mod}(A)}\int_A \rho^2(z)dxdy \\ &\leq \frac{2\pi}{\mathrm{mod}(A)}\int_B \rho^2(z)dxdy.\end{aligned}$$

So $\frac{2\pi}{\mathrm{mod}(B)} \leq \frac{2\pi}{\mathrm{mod}(A)}$ which implies our desired result.

The next proposition connects K-quasiconformal maps to ring domains.

Proposition 3.48. *If $\phi : U \to V$ is K-quasiconformal and $A \subset\subset U$ is a ring domain, then we have*

$$\mathrm{mod}\big(\phi(A)\big) \leq K\mathrm{mod}(A).$$

Proof. By composing ϕ with a conformal map guaranteed by Theorem 3.42, we may assume $A = A_{1,R}$ where $R > 1$. If we let Q_1 be the upper Jordan quadrilateral with vertices $-R, -1, 1, R$ and Q_2 the lower (where b-sides are $(-R,-1)$ and $(1,R)$), we see by Definition 3.43 we have

$$\frac{1}{\mathrm{mod}(Q_1)} + \frac{1}{\mathrm{mod}(Q_2)} = \frac{2\pi}{\mathrm{mod}(A)}. \tag{3.49}$$

We claim that

$$\frac{1}{\mathrm{mod}\left(\phi(Q_1)\right)} + \frac{1}{\mathrm{mod}\left(\phi(Q_2)\right)} \leq \frac{2\pi}{\mathrm{mod}\left(\phi(A)\right)}. \tag{3.50}$$

Indeed, suppose f is a conformal map sending $\phi(A)$ to an annulus $A_{1,R}$. Then for any rectifiable γ in A connecting the disjoint boundary curves with $\rho := |f'/f|$, we have

$$\int_\gamma \rho(z)|dz| = \int_\gamma \frac{|f'(z)|}{|f(z)|}|dz| = \int_{f\circ\gamma}\frac{1}{|z|}|dz| \geq \int_{f\circ\gamma}\frac{1}{|z|}d|z| = \log R = \mathrm{mod}\left(\phi(A)\right).$$

So by the second statement of Proposition 3.11, we have

$$\frac{1}{\mathrm{mod}\left(\phi(Q_j)\right)} \leq \frac{1}{\mathrm{mod}\left(\phi(A)\right)^2}\int_{\phi(Q_j)} \rho^2(z)dxdy$$

for $j = 1, 2$. Thus

$$\begin{aligned}\frac{1}{\operatorname{mod}\big(\phi(Q_1)\big)} + \frac{1}{\operatorname{mod}\big(\phi(Q_2)\big)} &\leq \frac{1}{\operatorname{mod}\big(\phi(A)\big)^2} \int_{\phi(A)} \rho^2(z)dxdy \\ &= \frac{1}{\operatorname{mod}\big(\phi(A)\big)^2} \int_{f\circ\phi(A)} \frac{1}{|z|^2} dxdy \\ &= \frac{1}{\operatorname{mod}\big(\phi(A)\big)^2} \int_{A_{1,R}} \frac{1}{r} drd\theta \\ &= \frac{2\pi}{\operatorname{mod}\big(\phi(A)\big)}.\end{aligned}$$

Combining 3.49 and 3.50 along with the K-quasiconformality of ϕ, we see that

$$\begin{aligned}\frac{2\pi}{K\operatorname{mod}(A)} &= \frac{1}{K\operatorname{mod}(Q_1)} + \frac{1}{K\operatorname{mod}(Q_2)} \\ &\leq \frac{1}{\operatorname{mod}\big(\phi(Q_1)\big)} + \frac{1}{\operatorname{mod}\big(\phi(Q_2)\big)} \\ &\leq \frac{2\pi}{\operatorname{mod}\big(\phi(A)\big)}.\end{aligned}$$

□

We next state and prove two important results.

Proposition 3.51. *If $\phi : \mathbb{C} \to V$ is K-quasiconformal, then $V = \mathbb{C}$.*

Proof. By contradiction, assume $V \neq \mathbb{C}$. Since ϕ is a homeomorphism, V is simply connected and thus we may assume $V = D$ (by composing ϕ with the Riemann map mapping V to D). By Definition 3.43, the modulus $\log(R)$ of $A_{1,R}$ goes to infinity as $R \to \infty$. Now if we apply Proposition 3.48 to ϕ^{-1}, we see that the modulus of the ring domain $\phi(A_{1,R})$ also goes to infinity as $R \to \infty$. But the inner boundary of this ring domain is fixed while the outer boundary is bounded, so it is contained in an annulus for all R, contradicting Remark 3.47. The fact that it maps the inner boundary to the inner boundary and similarly for the outer boundary follows from the properness of ϕ. □

Theorem 3.52 (Grötszch modulus theorem)**.** *Let $0 < r < 1$ and $G := D \setminus [0, r]$. If $B \subset\subset D$ is a ring domain whose inner boundary encloses both 0 and r, then*

$$\operatorname{mod}(B) \leq \operatorname{mod}(G).$$

Note G itself is not a ring domain, but it is conformal to one, so its modulus is defined. One can see this by applying $\frac{1}{z}$ to G, translating $\frac{1}{r}$ to 0, and then applying $\sqrt{z}$ with a Cayley transformation.

Proof. This follows directly from Remark 3.47 after following the ring domain B by the maps described above. □

We next prove the quasiconformal version of Montel's theorem.

Theorem 3.53 (Montel)**.** *Let $\phi_n : U \to \phi_n(U)$ be a sequence of K-quasiconformal maps for some $K \geq 1$ where the $\phi_n(U)$ are uniformly bounded. Then there exists a subsequence that converges locally uniformly.*

Proof. Since we have uniform boundedness, we know by the Arzelà-Ascoli theorem (see page 222 of [11]) that we just need to show $\{\phi_n\}_{n\geq 1}$ is uniformly equicontinuous on every compact set $E \subset U$. By scaling, we may assume $\phi(U_n) \subseteq \frac{1}{2}D$. So if $E \subset U$ is an arbitrary compact set, let $r = \frac{1}{3}d(E, \partial U)$ with $E \subseteq \bigcup_{z\in E} B_r(z)$ an open cover. By compactness, there exists a finite subcover $\bigcup_{i=1}^{n} B_r(z_i)$. So if $\delta < r$, then $|z - w| < \delta$ implies that there exists $i = 1, \ldots, n$ such that $z \in B_r(z_i)$ and $w \in B_{2r}(z_i)$. Then for all $n \geq 1$ we have by K-quasiconformality that

$$\begin{aligned}
\frac{1}{K}\log\Big(\frac{2r}{\delta}\Big) &\leq \frac{1}{K}\operatorname{mod}\Big(\frac{z+w}{2} + A_{\delta,2r}\Big) \\
&\leq \operatorname{mod}\bigg(\phi_n\Big(\frac{z+w}{2} + A_{\delta,2r}\Big)\bigg) \\
&\leq \operatorname{mod}\bigg(D \setminus \Big[0, \frac{|\phi_n(z) - \phi_n(w)|}{|1 - \overline{\phi_n(w)}\phi_n(z)|}\Big]\bigg)
\end{aligned}$$

where the last inequality follows from Theorem 3.52 and applying an automorphism of the disk. If we define $m(x) = \operatorname{mod}(D \setminus [0, x])$, then by Theorem 3.52 we see that it is a decreasing function on $(0, 1)$ where $m(x) \to \infty$ as $x \searrow 0$. Thus we have

$$\frac{1}{K}\log\Big(\frac{2r}{\delta}\Big) \leq \operatorname{mod}\big(D \setminus [0, \tfrac{1}{2}|\phi_n(z) - \phi_n(w)|]\big) \tag{3.54}$$

for all $n \geq 1$. Thus given $\varepsilon > 0$, we can make $\delta < r$ small enough so as to make $m(\frac{1}{2}|\phi_n(z) - \phi_n(w)|)$ in (3.54) large enough to imply $|\phi_n(z) - \phi_n(w)| < \varepsilon$ for all $n \geq 1$. $\square$

4. Approximating Riemann Mappings

We return to circle packings, but in order to understand finite circle packing more fully, we need to consider infinite circle packings. The following is an example of an infinite circle packing that we will use throughout this section.

Definitions 4.1. The *regular hexagonal circle packing* (or *honeycomb packing* for short) is the collection of circles

$$\mathcal{H} := (z + S^1)_{z \in \Gamma}$$

where Γ is the hexagonal lattice

$$\Gamma := \{2n + 2e^{\frac{2\pi i}{3}} m \mid n, m \in \mathbb{Z}\}$$

and $z + S^1 := \{z + e^{i\theta} \mid \theta \in [0, 2\pi)\}$. This circle packing comes with the counterclockwise orientation induced by the plane. Note that two circles $z + S^1, w + S^1$ in this packing are tangent if and only if $z - w = 2e^{\frac{\pi}{3}k}$ for some $k = 1, 2, \ldots, 5$. Between any three mutually tangent circles in this packing is an open region which we define as the *interstice*. This interstice is inscribed in what we call the *dual circle* formed by the three points of tangency. The interstice can be viewed as a hyperbolic triangle in the dual circle, and by a calculation, we see that the radius of any dual circle is $1/\sqrt{3}$.

The next definition will be useful in the next two important lemmas from Rodin and Sullivan.

Definition 4.2. Circles $C_1, \ldots, C_n$ form a *chain* if they have disjoint interiors and C_i is externally tangent to C_{i+1} in a counterclockwise order for $i = 1, \ldots, n-1$. We call this chain *closed* if C_n is externally tangent to C_1. Given a circle C, an *external C-chain* $C_1, \ldots, C_n$ is a closed chain for which each circle in the chain is externally tangent to C, and similarly, an *internal C-chain* $C_1, \ldots, C_n$ is a closed chain for which each circle in the chain is internally tangent to C.

Lemma 4.3 (Ring Lemma)**.** *Let C be a circle of radius r with an external C-chain $C_1, C_2, \ldots, C_n$. Then there is a constant c_n only depending on n where the radii of each C_i is at least rc_n.*

Sketch of Proof. By scaling and translating, we may assume $C = S^1$. If $n = 3$, I claim that all the circles have radius greater than 1. Indeed, if $C_1 = 2 + S^1$, then $C_2 \subset \{y \geq 1\} \cup \{y > |x - 1|\}$ and $C_3 \subset \{y \leq -1\} \cup \{y < -|x - 1|\}$. But then $C_2 \cap C_3 = \varnothing$ and hence cannot be tangent. For the general case, by considering the polygon surrounding C, we see that there is a radius greater than π/n. Without loss of generality, we suppose C_1 has the largest radius. Now the radius C_2 cannot be too small because if it was, the next $n - 3$ circles would be forced in the cusp between C and C_1, which would then reduce to the $n = 3$ case. A similar argument can be given for the rest of the circles. $\square$

A rigorous proof of this lemma with an explicit calculation of c_n is given in Appendix B of [9]. This uses Decartes' circle theorem.

Lemma 4.4 (Honeycomb Length-Area Lemma)**.** *Let $n \geq 1$ and define $\mathcal{H}_n := \bigcup_{j=1}^{n} (C_i^j)_{i=1}^{6j}$ where $(C_i^1)_{i=1}^{6}$ is the external S^1-chain in $\mathcal{H}$, $(C_i^2)_{i=1}^{12}$ is the chain immediately afterwards, and so on. If $\mathcal{C}_n$ is a circle packing with the same nerve as*

$\mathcal{H}_n$ contained in a disk of radius R, then the circle C_0 in $\mathcal{C}_n$ corresponding to S^1 in $\mathcal{H}_n$ has radius at most $3\frac{R}{\sqrt{\log(n)}}$.

Proof. Let $r_i^j > 0$ be the radius of the corresponding circle in $\mathcal{C}_n$ corresponding to $C_i^j \in \mathcal{H}_n$. Then by definition of $\mathcal{C}_n$, we have

$$\sum_{j=1}^{n}\sum_{i=1}^{6j}\pi(r_i^j)^2 \leq \pi R^2.$$

Since $\sum_{j=1}^{n}\frac{1}{j} > \log(n)$, there exists $j = 1, \ldots, n$ such that

$$\sum_{i=1}^{6j}(r_i^j)^2 \leq \frac{R^2}{j\log(n)} \implies 6j\sum_{i=1}^{6j}(r_i^j)^2 \leq \frac{6R^2}{\log(n)}.$$

By the Cauchy-Schwarz inequality,

$$\sum_{i=1}^{6j} r_i^j \leq \frac{3R}{\sqrt{\log(n)}}.$$

Note that the left hand side is the semiperimeter of a polygon surrounding C_0, so after dividing by π, we have our desired bound. □

For next technical lemma, we first introduce some notations.

Notations 4.5. For any $z_0 + S^1$ in $\mathcal{H}$, we can define the *inversion map* $\iota_{z_0} : \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$ across this circle on the Riemann sphere by

$$\iota_{z_0}(z_0 + re^{i\theta}) := z_0 + \frac{1}{r}e^{i\theta}$$

for $0 < r < \infty$ and $\iota_{z_0}(z_0) := \infty$, $\iota_{z_0}(\infty) := z_0$. In other words,

$$\iota_{z_0}(z) := \overline{z - z_0}^{-1} + z_0.$$

Let G be the group of transformations from $\mathbb{C}\cup\{\infty\}$ to $\mathbb{C}\cup\{\infty\}$ generated by all the inversions ι_{z_0} of circles in $\mathcal{H}$. If I is the union of all the interstices in $\mathcal{H}$, then we define

$$GI := \bigcup_{g\in G} g(I)$$

as the union of the images of the interstitial regions I under all of the transformations in G.

Lemma 4.6. *$m(\mathbb{C}\setminus GI) = 0$ where m denotes the Lebesgue measure.*

Proof. Let $G\mathcal{H}$ denote all the circles formed by applying an element of G to the circles in $\mathcal{H}$. If $z \in \mathbb{C}\setminus GI$, then it lies in one of the circles in $\mathcal{H}$, and then after inverting through that circle, it lies in another circle in $\mathcal{H}$, and so forth. After undoing the inversions, we see that z lies in infinitely many circles in $G\mathcal{H}$. Let C be one of these circles. By definition of GI, GI contains a union of six interstices bounded by C that appear as the outline of an internal C-chain of six circles. Applying the same argument used to establish Lemma 4.3, we see that the six circles in this internal C-chain have radius comparable to C, and hence there exists $c > 0$ independent of C such that

$$m(GI \cap D_C) > c \cdot m(D_C)$$

where D_C is the disk enclosed by C. In particular, this c is the same for each circle in the nested sequence containing z. Since these circles' radii decrease to zero geometrically by Lemma 4.3, we see that the density of z satisfies

$$\Theta^2(\mathbb{C} \setminus GI, z) := \lim_{r \searrow 0} \frac{m\big(B_r(z) \cap \mathbb{C} \setminus GI\big)}{m\big(B_r(z)\big)} \leq 1 - c < 1.$$

But Lebesgue's density theorem says $\Theta^2(\mathbb{C} \setminus GI, z) = 1$ for almost all $z \in \mathbb{C} \setminus GI$, so we have our desired result. □

We next state and prove a quick lemma before a big theorem.

Lemma 4.7. *Let C_1, C_2, C_3 and C_1', C_2', C_3' be two chains. Then there exists a Möbius transformation ϕ that maps each C_i to C_i' and the interstice of C_1, C_2, C_3 conformally onto the interstice of C_1', C_2', C_3'.*

Proof. By first inverting the point of intersection of C_1, C_2, we can then apply scaling and rigid motions to normalize to the configuration $\{y = \pm 1\} \cup S^1$ where the interstitial region is mapped to $\{-1 < y < 1\} \cap \{x > 0\} \cap \{x^2 + y^2 > 1\}$. We call this transformation ϕ_1 and we let ϕ_2 the corresponding transformation for C_1', C_2'. We see $\phi := \phi_2^{-1} \circ \phi_1$ is our desired transformation. □

Theorem 4.8 (Rigidity of the Honeycomb Packing)**.** *Let $\mathcal{C}$ be an infinite, oriented circle packing in $\mathbb{C}$ with the same nerve as the honeycomb packing $\mathcal{H}$. Then $\mathcal{C}$ is equal to the honeycomb packing up to affine transformations and reflections.*

Proof. Step 1: By applying a reflection, we may assume $\mathcal{C}$ and $\mathcal{H}$ have the same orientation. For each interstice I_j of $\mathcal{H}$, there is an associated interstice I_j' of $\mathcal{C}$. By Lemma 4.7, there is a Möbius transformation $T_j : I_j \to I_j'$. These maps can be combined to form a map ϕ_0 that is conformal on the union of all the interstices of $\mathcal{H}$, which we denote I.

Step 2: We would eventually like to extend it to the entire complex plane by first defining it on the (punctured) disks enclosed by circles in $\mathcal{H}$. We first do this on $D \setminus \{0\}$, as defining ϕ_0 on the other disks is similar. Note S^1 in $\mathcal{H}$ is bounded by six interstices $I_1, \ldots, I_6$ which map to six other interstices $I_1', \ldots, I_6'$ that bound the corresponding circle C_0 and are bounded by an external C_0-chain $C_1, \ldots, C_6$. By Lemma 4.3, all of the circles $C_1, \ldots, C_6$ have radii comparable (both above and below) to the radius of C_0. As a consequence, using compactness and continuity of the coefficients in Möbius transformations, the map ϕ_0, viewed as a piecewise Möbius map mapping S^1 to C_0, has derivative comparable to the radius of C_0 also. We extend ϕ_0 on $D \setminus \{0\}$ by

$$\phi_0(re^{i\theta}) = A\big(r \cdot \phi_0(e^{i\theta})\big)$$

where $0 < r < 1$ and A is an affine transformation mapping S^1 to C_0. By Corollary 3.25, we see ϕ_0 is K-quasiconformal on $D \setminus \{0\}$ for some K. By extending ϕ_0 to the other punctured disks in $\mathcal{H}$ in the same fashion, we see these K are uniformly bounded above (which we abusively call K), and by many applications of Theorem 3.39, ϕ_0 is now K-quasiconformal on $\mathbb{C}$ and conformal on I. By Proposition 3.51, $\phi_0(\mathbb{C}) = \mathbb{C}$, and thus the circle packing $\mathcal{C}$ and all of its interstices cover the entire complex plane.

Step 3: We next use the circular version of the Schwarz reflection principle to replace ϕ_0 by another K-quasiconformal map that is conformal on a larger region than I. Let $z_j + S^1$ be an arbitrary circle in $\mathcal{H}$, and let C_{z_j} be the corresponding circle in $\mathcal{C}$. If ι_{z_j} and $\widetilde{\iota}_{z_j}$ are the inversions across $z_j + S^1$ and C_0 respectively, we let

$$\phi_1(z) := \widetilde{\iota}_{z_j} \circ \phi_0 \circ \iota_{z_j}(z)$$

for $z \in z_j + D$ and $\phi_1 \equiv \phi_0$ on or outside $z_j + S^1$. By Theorem 3.39, ϕ_1 is still K-quasiconformal. Also, it remains conformal on I, but is now also conformal on an additional interstitial region inside the disks of $\mathcal{H}$. Repeating this construction, one can find a sequence $\phi_n : \mathbb{C} \to \mathbb{C}$ of K-quasiconformal maps that map each circle $z_j + S^1$ to their counterparts C_0 and which are conformal on a sequence of sets I_n that increase up to GI. By Theorem 3.53, the restriction of ϕ_n to any compact set forms a normal family (uniform boundedness follows from the fact that these maps map $z_j + S^1$ to C_{z_j}). Using a diagonalization argument, the ϕ_n themselves are a normal family (and similarly for ϕ_n^{-1}). By passing to a subsequence, we may assume ϕ_n converge locally uniformly to a limit ϕ, and that ϕ_n^{-1} also converge locally uniformly to a limit that inverts ϕ. So ϕ is a homeomorphism that happens to be K-quasiconformal by Theorem 3.32. It is conformal on GI, and hence by Proposition 3.29 and Lemma 4.6, it is conformal. So $\phi \in \mathrm{Aut}(\mathbb{C}) = \{az + b \mid a, b \in \mathbb{C}\}$. ☐

Corollary 4.9 (Approximate Rigidity of the Honeycomb Packing)**.** *For any $\varepsilon > 0$, there exists $n > 0$ such that*

$$1 - \varepsilon \leq \frac{r_1}{r_0} \leq 1 + \varepsilon$$

where r_0 is the radius of the circle C_0 in $\mathcal{C}_n$ associated to S^1 in $\mathcal{H}_n$ and r_1 is the radius of an adjacent circle C_1 of C_0 ($\mathcal{C}_n$ and $\mathcal{H}_n$ are as in Lemma 4.4 where $R = 1$).

Proof. After we normalize $r_0 = 1$ and $C_0 = S^1$, we suppose the claim failed for contradiction. Then there exists a sequence n tending to infinity where r_1^n, the radius of C_1^n which is adjacent to $C_0^n = S^1$ in $\mathcal{C}_n$, stays away from 1. By many applications of Lemma 4.3, for each circle $z + S^1$ in $\mathcal{H}$, the corresponding circle C_z^n in $\mathcal{C}_n$ has radius bounded below by 0 and bounded above by 1. Passing to a subsequence using Bolzano-Weierstrass and using the Arzelà-Ascoli diagonalization argument, we may assume $r_z^n \to r_z^\infty > 0$ as $n \to \infty$. By applying a rotation, we may assume the circles C_1^n converge to a limit circle C_1^∞ (in the Hausdorff sense), and we may assume the orientation of $\mathcal{C}_n$ does not depend on n. Induction shows C_z^n converges to a limiting circle C_z^∞, giving a circle packing $\mathcal{C}_\infty$ with the same nerve as $\mathcal{H}$. But Theorem 4.8 guarantees $\mathcal{C}_\infty$ is an affine copy of $\mathcal{H}$, which, among other things, implies $r_1^\infty = r_0^\infty = 1$. Thus $r_1^n \to 1$, contradicting our initial assumption. ☐

We are now set to prove the main theorem of this thesis. Let U be a bounded, simply connected open set in $\mathbb{C}$ with two distinct points $z_0, z_1 \in U$. By Theorem 3.1, there is a unique conformal map $\phi : U \to D$ such that $\phi(z_0) = 0$ and $\phi(z_1) > 0$. We wish to approximate this map, but first need to define some notions that will aid in the proof.

Definitions 4.10.

- For $\varepsilon > 0$, we let $\varepsilon \cdot \mathcal{H}$ be the infinite honeycomb packing scaled by ε. For every circle in $\varepsilon \cdot \mathcal{H}$, we define the *flower* to be the union of the closed disk enclosed by the circle, the six interstices bounding it, and the six closed disks tangent to the circle.

- Let ε be small enough so that z_0 lies in a flower of a circle, say C_0, in $\varepsilon \cdot \mathcal{H}_n$, and this flower lies in U. Let $\mathcal{I}_\varepsilon$ be the set all circles in $\varepsilon \cdot \mathcal{H}$ that can be reached by a finite chain of consecutively tangent circles in $\varepsilon \cdot \mathcal{H}_n$ whose flowers all lie in U. Elements of $\mathcal{I}_\varepsilon$ are called *inner circles*, and the circles that are not in $\mathcal{I}_\varepsilon$ but are tangent to inner circles are called *border circles*. Note that because U is simply connected, the union of all the flowers of inner circles is also simply connected. Therefore, one can traverse the border circles by a closed chain of consecutively tangent circles, with the inner circles enclosed by this chain.
- Let $\mathcal{C}_\varepsilon$ be the circle packing consisting of inner and border circles. Applying Theorem 2.12, one can find a circle packing $\mathcal{C}'_\varepsilon$ in D with the same nerve (and orientation) as $\mathcal{C}_\varepsilon$ such that all the circles associated to border circles are internally tangent to S^1. Applying an automorphism of the disk, we may assume the flower containing z_0 in $\mathcal{C}_\varepsilon$ is mapped to the flower containing 0 in $\mathcal{C}'_\varepsilon$, and the flower containing z_1 is mapped to a flower containing a positive real (from the lemma proceeding the statement of the main theorem, z_1 will lie in such a flower for ε small enough). Let U_ε be the union of all the solid equilateral triangles formed by the nerve of $\mathcal{C}_\varepsilon$, and let D_ε be the corresponding union of all the solid equilateral triangles formed by the nerve of $\mathcal{C}'_\varepsilon$

Before the main theorem of this thesis, we prove two lemmas about convergence.

Lemma 4.11. *U_ε converges to U in the Hausdorff sense (In particular, $z_1 \in U_\varepsilon$ for sufficiently small ε).*

Proof. Before we prove this for U_ε, we first show a similar statement. Let δ be small enough such that $\operatorname{dist}(z_0, \partial U) > \delta$, and define open sets

$$V_\delta := \{z \in U \mid \operatorname{dist}(z, \partial U) > \delta, z \text{ and } z_0 \text{ are in the same connected component of } V_\delta\}.$$

We claim

$$\bigcup_{\delta>0} V_\delta = U.$$

Indeed, if $z \in U$, we connected it to z_0 via a path γ in U. This path is compact, the function $\operatorname{dist}(z, \partial U)$ has a minimum δ_0 on γ. Thus every point in γ is in $V_{\delta_0/2}$, in particular, $z \in V_{\delta_0/2}$.

We claim that V_δ converges to U in the Hausdorff sense as $\delta \to 0$. Let $\varepsilon > 0$. Denote $(\cdot)_\varepsilon$ the ε-neighborhood of a set. Since $\overline{U}$ is compact, the cover

$$\overline{U} \subseteq (U)_\varepsilon = \bigcup_{\delta>0} (V_\delta)_\varepsilon$$

has a finite subcover,

$$U \subseteq (V_{\delta_1})_\varepsilon \cup \cdots \cup (V_{\delta_n})_\varepsilon = (V_{\min\{\delta_1,\ldots,\delta_n\}})_\varepsilon.$$

So picking $\delta < \min\{\delta_1, \ldots, \delta_n\}$, we see $U \subseteq (V_\delta)_\varepsilon$. On the other hand, it is clear that $V_\delta \subseteq U \subseteq (U)_\varepsilon$. So we have shown $V_\delta \to U$ in the Hausdorff distance as $\delta \to 0$.

Now we claim $V_{7\varepsilon} \subseteq U_\varepsilon$. Indeed, let $z \in V_{7\varepsilon}$ and consider $\varepsilon \cdot \mathcal{H}$ on the entire plane. By construction of $\varepsilon \cdot \mathcal{H}$, z is within 2ε of the c_0 center of the nearest circle of $\varepsilon \cdot H$. The flower of the circle contains z and is contained in $B_{3\varepsilon}(c_0)$.

The union of the flowers of the circles in the flower of c_0 is contained in $B_{5\varepsilon}(c_0)$. Since $\operatorname{dist}(z, \partial U) > 7\varepsilon$ and $|z - c_0| < 2\varepsilon$, we see all of these circles mentioned are contained in

$$B_{5\varepsilon}(c_0) \subseteq B_{7\varepsilon}(z) \subseteq U.$$

So z is contained in the flower of an inner circle, and every circle in this flower is in inner circle. We can apply this argument on every point on a path connecting z and z_0 in $V_{7\varepsilon}$, so we have $z \in U_\varepsilon$.

Finally, since $V_{7\varepsilon} \subseteq U_\varepsilon \subseteq U$ and $V_{7\varepsilon} \to U$ in Hausdorff distance as $\varepsilon \to 0$, we have $U_\varepsilon \to U$ in Hausdorff distance as $\varepsilon \to 0$. □

Lemma 4.12. *D_ε converges to D in the Hausdorff sense.*

Proof. We first show that the radii of the circles in D_ε tend to 0 uniformly as $\varepsilon \to 0$. Indeed, let $\eta > 0$ and let n be large enough so that $\frac{3}{\sqrt{\log n}} < \eta$. Now let $\delta > 0$ be small enough such that if $\varepsilon < \delta$, then there exists a circle packing $\mathcal{P}_\varepsilon$ as a subset of $\mathcal{C}'_\varepsilon$ centered at 0 with the same nerve as $\mathcal{H}_{2n}$ (defined in Lemma 4.4). By construction, every circle in $\mathcal{H}_n \subset \mathcal{P}_\varepsilon$ is the center of a circle packing with the same nerve as $\mathcal{H}_n$. By Lemma 4.4, the radii of these circles are bounded above by

$$\frac{3}{\sqrt{\log n}} < \eta.$$

For each other circle C of $\mathcal{C}'_\varepsilon$, these exists a sequence of disjoint chains $(C_i^1)_{i=1}^{k_1}, \ldots, (C_i^n)_{i=1}^{k_n}$, each such that

- $(C_i^j)_{i=1}^{k_j}$ separates C from the origin and from a point on the unit sphere
- $k_j \leq 6j$.

Let r_{ij} be the radius of the circle C_i^j. Then by Cauchy-Schwarz, we have

$$\Big(\sum_{i=1}^{k_j} r_{ij}\Big)^2 \leq k_j \sum_{i=1}^{k_j} r_{ij}^2.$$

Let $\ell_j \coloneqq 2\sum_{i=1}^{k_j} r_{ij}$ denote the length of chain $(C_i^j)_{i=1}^{n_j}$. Then the above gives

$$\frac{\ell_j^2}{n_j} \leq 4\sum_{i=1}^{k_j} r_{ij}^2 \implies \sum_{j=1}^{n} \frac{\ell_j^2}{k_j} \leq 4\sum_{j=1}^{n}\sum_{i=1}^{k_j} r_{ij}^2 \leq 4.$$

If $\ell \coloneqq \min\{\ell_1, \ldots, \ell_n\}$, we have

$$\frac{\ell}{2} \leq \frac{1}{\sqrt{\frac{1}{k_1} + \cdots + \frac{1}{k_n}}} \leq \frac{1}{\sqrt{\sum_{k=1}^{n} \frac{1}{6k}}} \leq \frac{3}{\sqrt{\log n}} < \eta.$$

The first property of the chains guarantees that the radius of C is less than $\ell/2$, proving the desired claim.

Now we show $D_\delta \to D$ in Hausdorff distance as $\delta \to 0$. Indeed, let $\varepsilon > 0$ and let $\delta > 0$ be small enough to guarantee the radii of D_δ are less than $\varepsilon/2$. Note that border circles of D_δ form an internal S^1-chain, with each circle having a radius less than $\varepsilon/2$. Connecting each center of these circles with a straight line segment to the centers of adjacent border circles, we obtain a closed curve in D_δ that is within ε of S^1. Since D_δ is simply connected, it contains the inner region of this curve. In particular, it contains $(1-\varepsilon)D$. Thus

$$(1-\varepsilon)D \subseteq D_\delta \implies D \subseteq (D_\delta)_\varepsilon.$$

On the other hand, it is clear that $D_\delta \subseteq D \subseteq (D)_\varepsilon$. So we have shown $D_\delta \to D$ in the Hausdorff distance as $\delta \to 0$. □

Theorem 4.13 (Rodin-Sullivan-Thurston Theorem). *Let U be a bounded, simply connected open set in $\mathbb{C}$ with two distinct points $z_0, z_1 \in U$. If $\phi_\varepsilon : U_\varepsilon \to D_\varepsilon$ be the piecewise affine map each triangle in U_ε to the associated triangle in D_ε. Then ϕ_ε converges locally uniformly to ϕ as $\varepsilon \to 0$ where ϕ is the unique conformal map from U to D such that $\phi(z_0) = 0$ and $\phi(z_1) > 0$, as guaranteed by Theorem 3.1.*

Proof. By Corollary 4.9, if we fix a compact set $K \subset U$, as $\varepsilon \to 0$ the circles in $\mathcal{C}'_\varepsilon$ corresponding to adjacent circles of $\mathcal{C}_\varepsilon$ in K have radii differing by a ratio of $1 + o(1)$. This implies that for any compact $K' \subset D$, adjacent circles of $\mathcal{C}'_\varepsilon$ in K' also have radii differing by a ratio of $1 + o(1)$. By basic trigonometry, the triangles of D_ε in K' are approximately equilateral (in the sense that each angle is $\frac{\pi}{3} + o(1)$). By Theorem 3.22, ϕ_ε is $1 + o(1)$-quasiconformal on the corresponding triangles, and so by Theorem 3.39, it is $1+o(1)$-quasiconformal on K. Note Theorem 3.53 guarantees every sequence of ϕ_ε has a subsequence which converges locally uniformly on U, and whose inverses converge locally uniformly on D. This limit is thus a homeomorphism from U to D that maps z_0 to 0 and z_1 to a positive real number. By Theorem 3.32, the limit is locally 1-quasiconformal and hence conformal (by Proposition 3.18). By uniqueness, it must equal ϕ. As ϕ is the unique limit point of all subsequences of the ϕ_ε, this implies that ϕ_ε converges locally uniformly to ϕ. □

Appendix

The proof presented obtains the Riemann map as an abstract supremum of a family of functions. This next proof realizes the Riemann map as the limit of compositions of expansions, expanding the domain U in Step 1 of the original proof to the entire disc. We begin with some definitions.

Definition 4.14. The *inner radius* of a region $U \subset D$ that contains the origin is defined by

$$r_U := \sup\{\rho > 0 : D(0,\rho) \subseteq U\}.$$

Lemma 4.15. *Given a region $U \subset D$ that contains the origin, we have*

$$D(0, r_U) \subseteq U.$$

Furthermore, there exists $z \in \partial U$ such that $r_U = |z|$.

Proof. For the first statement, suppose $z \in D(0, r_U)$. Then since $d := \frac{|z|+r_U}{2} < r$, by definition of supremum, we have $z \in D(0,d) \subseteq U$. For the second statement, by definition of supremum, there exists $z_1 \in D(0, r_U + 1)$ with $z_1 \notin U$. Similarly, there exists $z_2 \in D(0, \min\{r_U + 1/2, |z_1|\})$ with $z_2 \notin U$. Continuing this way, we obtain a bounded sequence z_n where $z_n \in D(0, \min\{r_U + 1/n, |z_{n-1}|\})$ with $z_n \notin U$ which has a convergent subsequence (which we also call z_n with limit point z). Since

$$r_U \leq |z_n| \leq r_U + \frac{1}{n},$$

we see $|z| = r_U$. As a limit of a sequence outside of U, we see $z \in \overline{\mathbb{C} \setminus U}$, but $z \in \overline{D(0, r_U)} \subseteq \overline{U}$, so $z \in \partial U$ as desired. $\square$

Now for the definition of expansion.

Definition 4.16. A holomorphic injection $f : U \to D$ (where U is a region that contains 0) is an *expansion* if $f(0) = 0$ and $|f(z)| > |z|$ for $z \in U - \{0\}$.

Lemma 4.17. *An expansion $f : U \to D$ has the properties $r_{f(U)} \geq r_U$ and $|f'(0)| > 1$.*

Proof. The second inequality is an easy consequence of the Riemann removable singularity theorem with the maximum modulus principle applied to $z/f(z)$. For the first statement, it suffices to show $D(0,\rho) \subseteq f\big(D(0,\rho)\big)$ for $0 < \rho < r_U$. Suppose there exists $w \in D(0,\rho)$ for which $f(z) \neq w$ for all $z \in D(0,\rho)$. Then

$$\frac{1}{f(z) - w}$$

is holomorphic on $D(0,\rho)$ and extends continuously to the boundary (this is because $|f(\rho e^{i\theta})| > \rho > |w|$ so $f(z) \neq w$ for $z \in \overline{D(0,\rho)}$). By the maximum modulus principle, we have

$$\begin{aligned}
\max_{z \in \overline{D(0,\rho)}} \frac{1}{|f(z) - w|} &= \max_{z \in \partial D(0,\rho)} \frac{1}{|f(z) - w|} \\
&\leq \max_{z \in \partial D(0,\rho)} \frac{1}{|f(z)| - |w|} \\
&\leq \max_{z \in \partial D(0,\rho)} \frac{1}{\rho - |w|}
\end{aligned}$$

which implies

$$\min_{z\in \overline{D(0,\rho)}} |f(z) - w| \geq \rho - |w|.$$

Thus $f\big(D(0, \rho)\big) \subseteq D - D(w, \rho - |w|)$. One can repeat this argument with finitely many points on the line segment between 0 and w to deduce that 0 is not in $f\big(D(0, \rho)\big)$, a clear contradiction with $f(0) = 0$. $\square$

Proof of Existence in Theorem 3.1 via expansions. Using step 1 in the previous proof of this theorem, we can assume U is a Koebe domain U, that is, an open, simply connected set in D that contains the origin. We wish to produce an expansion $f : U \to D$. Indeed, let $\alpha \in \partial U$ be such that $|\alpha| = r_U$, which can be done by lemma 4.15. If

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z},$$

we see that ψ_α is nonzero on U, and so there exists $h : U \to D$ such that

$$h^2 = \psi_\alpha$$

on U. We claim $f : U \to D$ defined by $f := \psi_{h(0)} \circ h$ is an expansion. It is clearly a holomorphic injection that fixes the origin. Note that $f^{-1} = \psi_\alpha \circ \psi_{h(0)}^2$ is defined on D and satisfies the conditions for Schwartz lemma. Since f^{-1} is not injective on D, we have $|f^{-1}(z)| < |z|$ on $D - \{0\}$, and in particular, we have $|z| < |f(z)|$ on $U - \{0\}$, as desired. Note that by the chain rule, we have

$$|f'(0)| = \frac{1 + r_U}{2\sqrt{r_U}}. \tag{4.18}$$

Now we define $U_0 := U$ and $f_0 : U_0 \to D$ an expansion as above. Note that $U_1 := f_0(U_0)$ is also a Koebe domain, so we can define the expansion $f_1 : U_1 \to D$ the same way. Inductively, we have $U_n := f_{n-1}(U_{n-1})$ and $f_n : U_n \to D$ an expansion. Define $F_n : U \to D$ by $F_n := f_n \circ \cdots \circ f_0$. Since F_n are uniformly bounded, there exists a subsequence (which we also call F_n) that converges to a function $F : U \to D$ uniformly on compact subsets of U. Note that F is an expansion because clearly $F(0) = 0$ and

$$|F(z)| \geq |F_1(z)| > |f_0(z)| > |z|$$

for $z \in U - \{0\}$. By a corollary of Hurwitz's Theorem, since F_n are injective, F must also be injective (for otherwise $F = 0$, a contradiction with the above inequality). We wish to show F is surjective. It suffices to show $r_{F(U)} \geq 1$. Indeed, note

$$|F_n'(0)| = \prod_{k=0}^{n} |f_k'(0)|$$

is a strictly increasing sequence greater than 1 (by Lemma 4.17), and it is bounded above by $1/r_U$ by the Schwartz lemma. So it converges, and after taking logarithms, we see $\lim_{n\to\infty} |f_n'(0)| = 1$. Using 4.18, we see $\lim_{n\to\infty} r_{F_n(U)} = 1$ (note $r_{F_n(U)}$ converges since is nondecreasing by Lemma 4.17 and it bounded above by 1). Note that for all n, there exists an expansion $G_n : U_{n+1} \to D$ such that $F = G_n \circ F_n$. So by Lemma 4.17, we have $r_{F(U)} \geq r_{F_n(U)}$ for all n, and hence $r_{F(U)} \geq 1$ as desired. $\square$

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This thesis is just the beginning. I am just getting started.

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